Warwick Mathematics Exchange

MA453

Lie Algebras

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Desync, aka The Big Ree

Contents

Introduction

A *Lie algebra* is a vector space equipped with an additional multiplication operation that is typically non-associative. Lie algebras are closely related to Lie groups, which are groups that are also smooth manifolds; every Lie group induces a Lie algebra as the tangent space at the identity, in which case, the Lie bracket measures the failure of commutativity for the Lie group.

Conversely, to any finite-dimensional Lie algebra over the $\mathbb{R}$ or $\mathbb{C}$, there is a corresponding connected Lie group. This correspondence allows us to study the structure and classification of Lie groups in terms of Lie algebras.

Lie groups and Lie algebras find extensive applications in physics – in particular, quantum and particle mechanics – where Lie groups arise as symmetry groups of physical systems and their Lie algebras may be interpreted as infinitesimal symmetry motions of those systems.

Disclaimer: I make *absolutely no guarantee* that this document is complete nor without error. In particular, any content covered exclusively in lectures (if any) will not be recorded here. This document was written during the 2023 academic year, so any changes in the course since then may not be accurately reflected.

Notes on formatting

New terminology will be introduced in *italics* when used for the first time. Named theorems will also be introduced in *italics*. Important points will be **bold**. Common mistakes will be underlined. The latter two classifications are under my interpretation. YMMV.

Content not taught in the course will be outlined in the margins like this. Anything outlined like this is not examinable, but has been included as it may be helpful to know alternative methods to solve problems.

The table of contents above, and any inline references are all hyperlinked for your convenience.

History

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Authors

This document was written by R.J. Kit L., a maths student. I am not otherwise affiliated with the university, and cannot help you with related matters.

Please send me a PM on Discord @Desync#6290, a message in the WMX server, or an email to Warwick.Mathematics.Exchange@gmail.com for any corrections. (If this document somehow manages to persist for more than a few years, these contact details might be out of date, depending on the maintainers. Please check the most recently updated version you can find.)

If you found this guide helpful and want to support me, you can buy me a coffee!

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*Storing dates in big-endian format is clearly the superior option, as sorting dates lexicographically will also sort dates chronologically, which is a property that little and middle-endian date formats do not share. See ISO-8601 for more details. This footnote was made by the computer science gang.

1 Lie Algebras

All of the vector spaces we consider will be finite dimensional over a field K.

A *Lie bracket* on a vector space L is a bilinear map $[-,-] : V \times V \to V$ with the additional properties:

(L1) (*Alternation*) For all $x \in L$, $[x,x] = 0$;

(L2) (*Jacobi identity*) For all $x,y,z \in L$, $\big[x,[y,z]\big] + \big[z,[x,y]\big] + \big[y,[x,z]\big] = 0$.

The pair $\big(L,[-,-]\big)$ is then called a *Lie algebra* over K. We often suppress the Lie bracket and the field, and refer to a Lie algebra by the underlying vector space L.

The *dimension* of a Lie algebra L is the dimenion of L as a vector space.

Lemma 1.1 (Anticommutativity)**.** *Let L be a Lie algebra. Then, for all $x,y \in L$,*

$$[x,y] = -[y,x]$$

Proof. By (L1), for all $x,y \in L$, $[x+y,x+y] = 0$, so by bilinearity, $[x,x]+[x,y]+[y,x]+[y,y] = 0$. Again by (L1), $[x,x] = 0 = [y,y]$, so $[x,y] + [y,x] = 0$, and hence $[x,y] = -[y,x]$. ■

Lemma 1.2. *If* $\operatorname{char}(K) \neq 2$*, then the alternating property is equivalent to anticommutativity.*

Proof. The forward implication is shown in the previous lemma. Conversely, suppose L satisfies (L2) and anticommutativity. Then, $[x,x] = -[x,x]$, so $2[x,x] = 0$. Since the characteristic of K is not 2, 2 is invertible, so $[x,x] = 0$. ■

Example.

(*i*) Let V be any vector space and define $[-,-] : V \times V \to V$ to be the constant zero vector map. This bracket trivially satisfies the Lie bracket axioms, so $\big(V,[-,-]\big)$ is a Lie algebra, called an *abelian* Lie algebra.

Every 1-dimensional Lie algebra is necessarily abelian since if e is the basis element, then $[a,b] = [\alpha e,\beta e] = \alpha\beta[e,e] = 0$.

(*ii*) Let $L = \mathbb{R}^3$ be a vector space over $\mathbb{R}$. The cross product satisfies the Lie bracket axioms, so $\mathbb{R}^3$ is a Lie algebra over $\mathbb{R}$.

(*iii*) Consider the set $L = M_n(K)$ of $n \times n$ matrices with entries in K as a n^2-dimensional vector space over K. Define the bracket $[-,-] : L \times L \to L$ by

$$[A,B] = AB - BA$$

This is linear in the first argument:

$$[\lambda A + \mu B,C] = (\lambda A + \mu B)C - C(\lambda A + \mu B) = \lambda(AC - CA) + \mu(BC - CB) = \lambda[A,C] + \mu[B,C]$$

and since $[A,B] = AB - BA = -(BA - AB) = -[B,A]$, we also have linearity in the second argument. The bracket is also alternating since $[A,A] = AA - AA = \mathbf{0}$. We also have:

$$\begin{aligned}
\big[A,[B,C]\big] &= [A,BC - CB] \\
&= A(BC - CB) - (BC - CB)A \\
&= ABC - ACB - BCA + CBA \\
\big[C,[A,B]\big] &= CAB - CBA - ABC + BAC \\
\big[B,[C,A]\big] &= BCA - BAC - CAB + ACB
\end{aligned}$$

The 12 terms are the positive and negatives of the permutations of A, B, and C, so adding these together, we obtain $\mathbf{0}$, and the Jacobi identity holds. So $\big(M_n(K),[-,-]\big)$ is a Lie algebra.

This Lie algebra is also denoted by $\mathfrak{gl}_n(K)$ (since it is the Lie algebra of the Lie group $GL_n(K)$).

(iv) Let V be any vector space and consider the endomorphism space $\mathrm{End}(V)$ of V. We similarly define the bracket $[-,-] : \mathrm{End}(V) \times \mathrm{End}(V) \to \mathrm{End}(V)$ by $[S,T] = S \circ T - T \circ S$. This defines a Lie algebra, denoted by $\mathfrak{gl}(V)$.

(v) Consider the linear subspace $L = \big\{A \in M_n(K) : \mathrm{tr}(A) = 0\big\} \subseteq M_n(K)$ of matrices with zero trace. Since the trace is linear and satisfies $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, the restriction of the Lie bracket from $\mathfrak{gl}_n(K)$ is closed on L since $\mathrm{tr}\big([A,B]\big) = \mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = \mathrm{tr}(AB) - \mathrm{tr}(AB) = 0$. Thus, L is again a Lie algebra, denoted by $\mathfrak{sl}_n(K)$ (again, since it is the Lie algebra of the Lie group $SL_n(K)$).

(vi) Let $L = \big\{A \in M_n(K) : a_{ij} = 0 \text{ for all } i > j\big\}$ be the set of **non-strictly** upper triangular matricies over K. Again, the Lie bracket from $\mathfrak{gl}_n(K)$ is closed on L since the product and sum of upper triangular matricies is again upper triangular, so L is again a Lie algebra, denoted by $\mathfrak{b}_n(K)$.

(vii) Let $L = \big\{A \in M_n(K) : a_{ij} = 0 \text{ for all } i \geq j\big\}$ be the set of **strictly** upper triangular matricies over K. Again, L is a Lie algebra, denoted by $\mathfrak{u}_n(K)$.

$\triangle$

1.1 Structure Constants

Let L be a Lie algebra, and let $e_1, \dots, e_n$ be a basis of L. Then, for $a,b \in L$, we may express them as linear combinations of the basis elements and use the linearity of the Lie bracket to obtain:

$$\begin{aligned} [a,b] &= \left[\sum_i \alpha_i e_j, \sum_j \beta_j e_j\right] \\ &= \sum_{i,j} \alpha_i \beta_j [e_i,e_j] \end{aligned}$$

Removing the diagonal elements and combining the antisymmmetric combinations, this simplifies to:

$$= \sum_{i,j} (\alpha_i \beta_j - \beta_i \alpha_j)[e_i,e_j]$$

If we compute the Lie brackets $[e_i,e_j]$ of the basis elements, then we can compute any other Lie bracket $[a,b]$ using this formula.

Example. Consider the space $\mathfrak{gl}_2(\mathbb{R})$ with basis

$$e_1 = E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad e_2 = E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad e_3 = E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad e_4 = E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Applying the Lie bracket to pairs of these basis elements, we have:

$$\begin{array}{llll} [e_1,e_1] = \mathbf{0} & [e_1,e_2] = e_2 & [e_1,e_3] = -e_3 & [e_1,e_4] = \mathbf{0} \\ [e_2,e_1] = -e_2 & [e_2,e_2] = \mathbf{0} & [e_2,e_3] = e_1 - e_4 & [e_2,e_4] = e_2 \\ [e_3,e_1] = e_3 & [e_3,e_2] = e_4 - e_1 & [e_3,e_3] = \mathbf{0} & [e_3,e_4] = -e_3 \\ [e_4,e_1] = \mathbf{0} & [e_4,e_2] = -e_2 & [e_4,e_3] = e_3 & [e_4,e_4] = \mathbf{0} \end{array}$$

(Since the Lie bracket is anticommutative and alternating, we only really need to compute the 6 entries above the diagonal.)

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1e_1 + 2e_2 + 3e_3 + 4e_4 \qquad B = \begin{bmatrix} 5 & -6 \\ -7 & 8 \end{bmatrix} = 5e_1 - 6e_2 - 7e_3 + 8e_4$$

By direct computation, the Lie bracket $[A,B]$ is given by:

$$\begin{aligned} [A,B] &= AB - BA \\ &= \begin{bmatrix} -9 & 10 \\ -13 & 14 \end{bmatrix} - \begin{bmatrix} -13 & -14 \\ 17 & 18 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 24 \\ -30 & -4 \end{bmatrix} \end{aligned}$$

Alternatively, the formula yields:

$$\begin{aligned} [A,B] &= \big(1\cdot(-6) - 5\cdot 2\big)[e_1,e_2] + (1\cdot(-7) - 5\cdot 3)[e_1,e_3] + (1\cdot 8 - 5\cdot 4)[e_1,e_4] \\ &\quad + \big(2\cdot(-7) - (-6)\cdot 3\big)[e_2,e_3] + \big(2\cdot 8 - (-6)\cdot 4\big)[e_2,e_4] + \big(3\cdot 8 - (-7)\cdot 4\big)[e_3,e_4] \\ &= -16[e_1,e_2] - 22[e_1,e_3] - 12[e_1,e_4] + 4[e_2,e_3] + 40[e_2,e_4] + 52[e_3,e_4] \\ &= -16e_2 + 22e_3 + 4(e_1 - e_4) + 40e_2 - 52e_3 \\ &= 4e_1 + 24e_2 - 30e_3 - 4e_4 \\ &= \begin{bmatrix} 4 & 24 \\ -30 & -4 \end{bmatrix} \end{aligned}$$

△

Since $[e_i,e_j] \in L$, these brackets themselves can also be expressed in the basis as:

$$[e_i,e_j] = \sum_k c_{ij}^k e_k$$

The coefficients c_{ij}^k are called the *structure constants* of L with respect to the basis $e_1,\dots,e_n$.

Example. In the above example, the Lie brackets are:

$$\begin{aligned} &[e_1,e_2] = e_2, && [e_1,e_3] = -e_3 && [e_1,e_4] = \mathbf{0} \\ &[e_2,e_3] = e_1 - e_4, && [e_2,e_4] = e_2 && [e_3,e_4] = -e_3 \end{aligned}$$

The corresponding structure constants are thus given by:

$$\begin{aligned} &c_{12}^1 = 0, && c_{12}^2 = 1, && c_{12}^3 = 0, && c_{12}^4 = 0, \\ &c_{13}^1 = 0, && c_{13}^2 = 0, && c_{13}^3 = -1, && c_{13}^4 = 0, \\ &c_{14}^1 = 0, && c_{14}^2 = 0, && c_{14}^3 = 0, && c_{14}^4 = 0, \\ &c_{23}^1 = 1, && c_{23}^2 = 0, && c_{23}^3 = 0, && c_{23}^4 = -1, \\ &c_{24}^1 = 0, && c_{24}^2 = 1, && c_{24}^3 = 0, && c_{24}^4 = 0, \\ &c_{34}^1 = 0, && c_{34}^2 = 0, && c_{34}^3 = -1, && c_{34}^4 = 0, \end{aligned}$$

In more detail, $[e_1,e_2] = 0e_1 + 1e_2 + 0e_3 + 0e_4$, so the four corresponding structure constants (the first row) are the coefficients 0, 1, 0, and 0. △

As a shortcut, the Lie bracket of elementary matrices may be computed as:

$$[E_{ij},E_{k\ell}] = \delta_{jk}E_{i\ell} - \delta_{\ell i}E_{kj}$$

1.2 Homomorphisms

Let L_1 and L_2 be Lie algebras over a field K. A function $\phi : L_1 \to L_2$ is a *Lie algebra homomorphism* if:

(*i*) ϕ is K-linear;

(*ii*) $\forall x,y \in L_1, \phi\big([x,y]_{L_1}\big) = \big[\phi(x),\phi(y)\big]_{L_2}$.

That is, a Lie algebra homomorphism is a homomorphism of the underlying vector space that preserves the Lie bracket.

Given a basis $\{e_i\}_{i=1}^n$ of L_1, it follows from the previous formula $[a,b] = \sum_{i,j} \alpha_i\beta_j[e_i,e_j]$ that, to show property (*ii*), it suffices to verify that ϕ preserves the Lie brackets $[e_i,e_j]$ of the basis elements.

If ϕ is furthermore bijective (or equivalently, invertible), then ϕ is a *Lie algebra isomorphism.* If there exists a Lie algebra isomorphism between L_1 and L_2, we say that L_1 and L_2 are isomorphic, and denote this relation by $L_1 \cong L_2$.

Lemma 1.3. *Let L_1 and L_2 be Lie algebras over a field K. Then, $L_1 \cong L_2$ if and only if there exist bases $\mathcal{B}_1$ of L_1 and $\mathbb{B}_2$ of L_2 such that the structure constants c_{ij}^k of L_1 with respect to $\mathcal{B}_1$ are the same as the structure constants of d_{ij}^k of L_2 with respect to $\mathcal{B}_2$.*

Proof. For the forward implication, let $\mathcal{B}_1 = (e_i)_{i=1}^n$ be a basis of L_1 and let $\phi : L_1 \to L_2$ be a Lie algebra isomorphism. Transport the basis $\mathcal{B}_1$ along ϕ to a basis $\mathcal{B}_2 = (f_i)_{i=1}^n = \big(\phi(e_i)\big)_{i=1}^n$ of L_2.

Then,

$$\begin{aligned}\phi\big([e_i,e_j]\big) &= \phi\left(\sum_k c_{ij}^k e_k\right)\\ &= \sum_k c_{ij}^k \phi(e_k)\\ &= \sum_k c_{ij}^k f_k\end{aligned}$$

We also have

$$\begin{aligned}\phi\big([e_i,e_j]\big) &= \big[\phi(e_i),\phi(e_j)\big]\\ &= [f_i,f_j]\\ &= \sum_k d_{ij}^k f_k\end{aligned}$$

Comparing coefficients, we have $c_{ij}^k = d_{ij}^k$ for all i,j,k.

For the reverse implication, suppose there exist bases $\mathcal{B}_1 = (e_i)_{i=1}^n$ of L_1 and $\mathcal{B}_2 = (f_i)_{i=1}^n$ of L_2 such that $c_{ij}^k = d_{ij}^k$ for all i,j,k.

Define a linear map $\phi : L_1 \to L_2$ on basis elements by $e_i \mapsto f_i$ and linearly extending. This is a K-linear isomorphism since $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases. It remains to check that ϕ is a Lie algebra homomorphism.

$$\begin{aligned}\phi\big([e_i,e_j]\big) &= \phi\left(\sum_k c_{ij}^k e_k\right)\\ &= \sum_k c_{ij}^k \phi(e_k)\\ &= \sum_k c_{ij}^k f_k\end{aligned}$$

$$\begin{aligned} &= \sum_k d_{ij}^k f_k \\ &= [f_i,f_j] \\ &= [\phi(e_i),\phi(e_j)] \end{aligned}$$

■

Example. For any Lie algebra L, the identity map $\mathrm{id}_L : L \to L$ is trivially a Lie algebra isomorphism. △

Example. For any field K, the trace map $\mathrm{tr} : \mathfrak{gl}_n(K) \to K$ is a Lie algebra homomorphism, where K is equipped with the identically zero Lie bracket (i.e. is abelian):

(i) trace is linear;

(ii) for all $A,B \in \mathfrak{gl}_n(K)$, we have $\mathrm{tr}\big([A,B]\big) = 0$ by basic properties of the trace, while $[\mathrm{tr}(A), \mathrm{tr}(B)] = 0$ since K is abelian. So the trace preserves the Lie bracket.

△

1.3 Subalgebras

Let L be a Lie algebra. A *Lie subalgebra* K of L is a subset $K \subseteq L$ such that:

(i) K is a linear subspace of L;

(ii) K is closed under the Lie bracket: $\forall a,b \in K, [a,b] \in K$.

That is, K is a subset of L that is also a Lie algebra under (the restriction of) the Lie bracket of L.

Example. $\mathfrak{sl}_n(K)$ (zero-trace matrices) is a Lie subalgebra of $\mathfrak{gl}_n(K)$ (all matrices). Similarly, $\mathfrak{b}_n(K)$ (upper triangular) and $\mathfrak{u}_n(K)$ (strictly upper triangular) are subalgebras of $\mathfrak{gl}_n(K)$.

Moreover, any strictly upper triangular matrix has zero trace, so $\mathfrak{u}_n(K) \subseteq \mathfrak{sl}_n(K)$, and every strictly upper triangular matrix is upper triangular, so also $\mathfrak{u}_n(K) \subseteq \mathfrak{b}_n(K)$. △

Example. Consider the space $\langle e_2,e_3\rangle$ in $\mathfrak{gl}_2(K)$. This is a linear subspace of $\mathfrak{gl}_2(K)$, but is not a Lie subalgebra, since it is not closed under the Lie bracket: $[e_2,e_3] = e_1 - e_4 \notin \langle e_2,e_3\rangle$. △

1.4 Ideals

Let L be a Lie algebra. An *ideal* I of L is a subset $I \subseteq L$ such that:

(i) I is a linear subspace of L;

(ii) I absorbs Lie brackets with any element of L: $\forall x \in L\, \forall i \in I, [x,i] \in I$.

Clearly, every ideal is a subalgebra, but the converse generally fails. Also, unlike for rings, there is no distinction between left, right, and two-sided ideals, since if $[x,i] \in I$, then $[i,x] = -[x,i] \in I$, as I is a linear subspace.

Example.

(i) $\mathfrak{sl}_n(K)$ is an ideal of $\mathfrak{gl}_n(K)$, since the trace of a Lie bracket is always zero.

(ii) Neither $\mathfrak{b}_n(K)$ nor $\mathfrak{u}_n(K)$ are ideals of $\mathfrak{gl}_n(K)$:

Let M_{ij} be the elementary matrix with $m_{ij} = 1$ and zero elsewhere.

For the former, let $A = M_{21} \in \mathfrak{gl}_n(K)$ and $B = M_{11} \in \mathfrak{b}_n(K)$. Then, $[A,B] = M_{21}M_{11} - M_{11}M_{21} = M_{21} - \mathbf{0} = M_{21}$ is not upper triangular.

For the latter, let $A = M_{21} \in \mathfrak{gl}_n(K)$ and $B = M_{112} \in \mathfrak{u}_n(K)$. Then, $[A,B] = M_{21}M_{12} - M_{12}M_{21} = M_{11} - M_{22}$ is diagonal, and not strictly upper triangular.

(*iii*) The previous counterexample for $\mathfrak{u}_n(K)$ also shows that $\mathfrak{u}_n(K)$ is not an ideal of $\mathfrak{sl}_n(K)$, since M_{21} in particular has zero trace.

(*iv*) $\mathfrak{u}_n(K)$ is an ideal of $\mathfrak{b}_n(K)$ since if $A \in \mathfrak{u}_n(K)$ and $B \in \mathfrak{b}_n(K)$, then the diagonals of AB and BA are zero, so $[A,B] \in \mathfrak{u}_n(K)$.

△

Lemma 1.4. *For any Lie algebra L,*

(*i*) *L is an ideal of L;*

(*ii*) *$\{0_L\}$ is an ideal of L;*

(*iii*) *The centre $Z(L) = \{z \in L : \forall x \in L, [z,x] = 0\}$ is an ideal of L.*

Proof.

(*i*) This is trivial since the Lie bracket is closed on L by definition.

(*ii*) $\{0_L\}$ is a linear subspace of L, and for any $x \in L$, $[x,0] = [x,0+0] = [x,0] + [x,0]$, so $[x,0] = 0$.

(*iii*) The centre is a linear subspace of L since if $\alpha \in K$: $0_L \in Z(L)$ since $[0,x] = 0$; if $z_1,z_2 \in Z(L)$, then $[z_1 + z_2,x] = [z_1,x] + [z_2,x] = 0 + 0 = 0$; and if $z \in Z(L)$ and $\lambda \in K$, $[\lambda z,x] = \lambda[z,x] = \lambda 0 = 0$.

Now, if $z \in Z(L)$ and $x \in L$, $[z,x] = 0$ so $[z,x] \in Z(L)$, as required.

■

Lemma 1.5. *Let $\phi : L_1 \to L_2$ be a Lie algebra homomorphism. Then,*

(*i*) *$\mathrm{im}(\phi)$ is a Lie subalgebra of L_2;*

(*ii*) *$\ker(\phi)$ is an ideal of L_1.*

Proof. From basic linear algebra, $\mathrm{im}(\phi)$ is a subspace of L_2 and $\ker(\phi)$ is a subspace of L_1.

(*i*) For any $x,y \in \mathrm{im}(\phi)$, there exist $x',y' \in L_2$ such that $x = \phi(x')$ and $y = \phi(y')$. Then, $[x,y] = \big[\phi(x'),\phi(y')\big] = \phi\big([x',y']\big) \in \mathrm{im}(\phi)$, so $\mathrm{im}(\phi)$ is closed under the Lie bracket and is hence a Lie subalgebra.

(*ii*) For any $z \in L_1$ and $x \in \ker(\phi)$, $\phi\big([z,x]\big) = \big[\phi(z),\phi(x)\big] = \big[\phi(z),0\big] = 0$, so $[z,x] \in \ker(\phi)$, as required.

■

Example. Consider the Lie algebra homomorphism $\mathrm{tr} : \mathfrak{gl}_n(K) \to K$. The image is all of K, since for $\alpha \in K$, $\alpha E_{1,1}$ has trace α; and the kernel is, by definition, $\mathfrak{sl}_n(K)$. So, by the previous lemma, $\mathfrak{sl}_n(K)$ is an ideal of $\mathfrak{gl}_n(K)$ (as we have already verified before). △

Let L be a Lie algebra and $I,J \subseteq L$ be ideals of L. We define the *ideal sum* as the pointwise sum:

$$I + J \coloneqq \{i + j : i \in I, j \in J\}$$

and the *ideal Lie bracket* as the subspace generated by all of the Lie brackets:

$$[I,J] \coloneqq \big\langle [i,j] : i \in I, j \in J \big\rangle$$

Lemma 1.6. *Let L be a Lie algebra and $I,J \subseteq L$ be ideals of L. Then, $I \cap J$, $I + J$, and $[I,J]$ are ideals of L.*

Proof. From basic linear algebra, the three sets are linear subspaces of L.

(*i*) If $x \in L$ and $i \in I \cap J$, then $[x,i] \in I$ since $i \in I$, and $[x,i] \in J$ since $i \in J$. So $[x,i] \in I \cap J$.

(*ii*) If $x \in L$ and $i \in I + J$, then $i = i' + j'$ for some $i' \in I, j' \in J$. Then, $[x,i] = [x,i' + j'] = [x,i'] + [x,j']$. Since I is an ideal, $[x,i'] \in I$, and similarly, $[x,j'] \in J$. So $[x,i] \in I + J$.

(*iii*) If $x \in L$ and $i \in [I,J]$, then i is some linear combination of Lie brackets $[i',j']$, so

$$\begin{aligned}[x,i] &= \left[x, \sum_{k=1}^{n} c_k [i_k,j_k]\right] \\ &= \sum_{k=1}^{n} c_k \big[x,[i_k,j_k]\big]\end{aligned}$$

It remains to show that the $\big[x,[i_k,j_k]\big]$ are in $[I,J]$. By the Jacobi identity, for any $x \in L$, $i \in I$, and $j \in J$,

$$\begin{aligned}\big[x,[i,j]\big] &= -\big[j,[x,i]\big] - \big[i,[j,x]\big] \\ &= \big[[x,i],j\big] - \big[i,[j,x]\big]\end{aligned}$$

Since I is an ideal, $[x,i] \in I$, and similarly, $[j,x] \in J$. So $\big[x,[i,j]\big] \in [I,J]$.

■

Example. For any Lie algebra L, $[L,L]$ is an ideal of L called the *derived subalgebra* of L. △

1.5 Adjoint Homomorphism

Let L be a Lie algebra. In particular, L is a vector space, so we may also consider the Lie algebra $\mathfrak{gl}_n(L)$ of K-linear maps $L \to L$. We define the *adjoint homomorphism* $\mathrm{ad} : L \to \mathfrak{gl}_n(L)$ by

$$\mathrm{ad}(x) = [x,-]$$

Note that $\mathrm{ad}(x)$ is an element of $\mathfrak{gl}_n(L)$ since $\mathrm{ad}(x)(y) = [x,y] \in L$, and $\mathrm{ad}(x)$ is linear:

$$\begin{aligned}\mathrm{ad}(x)(\lambda y_1 + \mu y_2) &= [x,\lambda y_1 + \mu y_2] \\ &= \lambda[x,y_1] + \mu[x,y_2] \\ &= \lambda\,\mathrm{ad}(x)(y_1) + \mu\,\mathrm{ad}(x)(y_2)\end{aligned}$$

Lemma 1.7. *The adjoint homomorphism is a Lie algebra homomorphism.*

Proof. First, ad is linear:

$$\begin{aligned}\mathrm{ad}(\alpha x + \beta y)(z) &= [\alpha x + \beta y,z] \\ &= \alpha[x,z] + \beta[y,z] \\ &= \big(\alpha\,\mathrm{ad}(x) + \beta\,\mathrm{ad}(y)\big)(z)\end{aligned}$$

and ad also preserves the Lie bracket:

$$\mathrm{ad}\big([x,y]\big)(z) = \big[[x,y],z\big]$$

$$\begin{aligned}
&= -\big[z,[x,y]\big] \\
&= \big[x,[y,z]\big] + \big[y,[z,x]\big] \\
&= \big[x,[y,z]\big] - \big[y,[x,z]\big] \\
&= \mathrm{ad}(x)\big(\mathrm{ad}(y)(z)\big) - \mathrm{ad}(y)\big(\mathrm{ad}(x)(z)\big) \\
&= \big(\mathrm{ad}(x) \circ \mathrm{ad}(y) - \mathrm{ad}(y) \circ \mathrm{ad}(x)\big)(z) \\
&= \big[\mathrm{ad}(x), \mathrm{ad}(y)\big](z)
\end{aligned}$$

■

Lemma 1.8. *The kernel of the adjoint homomorphism is the centre of L:*

$$\ker(\mathrm{ad}) = Z(L)$$

Proof. By definition, the centre is the collection of all $z \in L$ such that $[z,x] = 0$ for all $x \in L$. That is, all the $z \in L$ such that $\mathrm{ad}(z)$ is the zero map, which is precisely the kernel of ad. ■

1.6 Quotient Algebras

The next standard algebraic construction is the quotient. As with rings, we will only obtain a Lie algebra when quotienting by an ideal.

Let V be a vector space, and $W \subseteq V$ a linear subspace. Given $v \in V$, we define the associated *coset* of W in V by:

$$v + W := \{v + w : w \in W\}$$

Lemma 1.9.

(*i*) *$x + W = y + W$ if and only if $x - y \in W$;*

(*ii*) *Any two cosets are either disjoint or equal.*

Proof.

(*i*) For forward implication, suppose $x+W = y+W$. Since $x = x+0 \in x+W = y+W$, $x = y+w$ for some $w \in W$. So $x - y = w \in W$. For the reverse implication, suppose $x - y = w_0 \in W$. Then, for any $x + w \in x + W$, $x + w = (y + w_0) + w = y + (w_0 + w) \in y + W$, so $x + W \subseteq y + W$. Similarly, for any $y + w \in y + W$, $y + w = (x - w_0) + w = x + (w - w_0) \in x + W$, so $y + W \subseteq x + W$. So $x + W = y + W$.

(*ii*) If the cosets are not disjoint, then there exists $z \in (x + W) \cap (y + W)$, so $y = x + w_1 = y + w_2$. Then, $x - y = w_2 - w_1 \in W$, so by the previous part, $x + W = y + W$.

■

We define the quotient V/W to be the set of cosets of V in W:

$$V/W := \{v + W : v \in V, w \in W\}$$

We define an addition $\oplus$ on V/W by:

$$(x + W) \oplus (y + W) := (x + y) + W$$

and a scalar multiplication by:

$$\lambda(x + W) := (\lambda x) + W$$

These are well-defined, as different representatives of the cosets differ only by an element of W, which is absorbed into the result. Under these operations, V/W inherits a vector space structure, with zero element $0_{V/W} = 0_V + W = W$.

Let L be a Lie algebra and I an ideal of L. We have that V/I is a vector space, but we can endow it with a bracket operation as follows:

$$[x + I, y + I] := [x, y] + I$$

This is well-defined since, if $x + I = x' + I$ and $y + I = y' + I$, then $x = x' + i_1$ and $y = y' + i_2$, and:

$$\begin{aligned} [x + I, y + I] &= [x, y] + I \\ &= [x' + i_1, y' + i_2] + I \\ &= [x', y'] + [x', i_2] + [i_1, x'] + [i_1, i_2] + I \end{aligned}$$

Since I is an ideal, the three bracket on the right are absorbed into I:

$$\begin{aligned} &= [x', y'] + I \\ &= [x' + I, y + I] \end{aligned}$$

Theorem 1.10. *This operation defines a Lie bracket on L/I.*

Proof.

(i) The bracket in L is bilinear, so the bracket in L/I is defined on representatives and thus inherits bilinearity:

$$\begin{aligned} \big[(\alpha x + I) \oplus (\beta y + I), z + I\big] &= \big[(\alpha x + \beta y) + I, z + I\big] \\ &= [\alpha x + \beta y, z] + I \\ &= \big(\alpha[x, z] + \beta[y, z]\big) + I \\ &= \alpha\big([x, z] + I\big) \oplus \beta\big([y, z] + I\big) \\ &= \alpha[x + I, z + I] \oplus \beta[y + I, z + I] \end{aligned}$$

and similarly in the second argument.

(ii) The bracket similarly inherits the alternating property:

$$\begin{aligned} [x + I, x + I] &= [x, x] + I \\ &= 0_L + I \\ &= 0_{L/I} \end{aligned}$$

(iii) The Jacobi identity also descends from the Lie bracket on L:

$$\begin{aligned} \big[x + I, [y + I, z + I]\big] &= [x, [y, z]] + I \\ \big[z + I, [x + I, y + I]\big] &= [z, [x, y]] + I \\ \big[y + I, [z + I, x + I]\big] &= [y, [z, x]] + I \end{aligned}$$

By the Jacobi identity in L, $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0_L$, so the sum of the three terms above reduces to $0_L + I = 0_{L/I}$.

∎

Let $\pi : L \to L/I$ be the natural quotient map $x \mapsto x + I$. Then,

$$\begin{aligned}\pi(x+y) &= (x+y)+I \\ &= (x+I) \oplus (y+I) \\ &= \pi(x) \oplus \pi(y)\end{aligned}$$

and

$$\begin{aligned}\pi(\lambda x) &= (\lambda x)+I \\ &= \lambda(x+I) \\ &= \lambda\pi(x)\end{aligned}$$

so π is linear; *pi* also preserves the Lie bracket:

$$\begin{aligned}\pi\big([x,y]\big) &= [x,y]+I \\ &= [x+I,y+I] \\ &= \big[\pi(x),\pi(y)\big]\end{aligned}$$

so π is a Lie algebra homomorphism.

Theorem 1.11 (First Isomorphism Theorem)**.** *Let $\phi : L_1 \to L_2$ be a Lie algebra homomorphism. Then,*

(*i*) $\mathrm{im}(\phi)$ *is a Lie subalgebra of L_2;*

(*ii*) $\ker(\phi)$ *is an ideal of L_1;*

(*iii*) $L_1/\ker(\phi) \cong \mathrm{im}(\phi)$

Proof. Parts (*i*) and (*ii*) were proved in Lemma 1.5.

For (*iii*), let $I = \ker(\phi)$ and define the map $f : L_1/I \to L_2$ by $f(x+I) = \phi(x)$. This is well-defined since if $x + I = y + I$, then $x - y \in I = \ker(\phi)$

$$\begin{aligned}f(x+I) &= \phi(x) \\ &= \phi(x-y+y) \\ &= \phi(x-y)+\phi(y) \\ &= 0+\phi(y) \\ &= \phi(y) \\ &= f(y+I)\end{aligned}$$

f is also linear, since

$$\begin{aligned}f\big((x+I)\oplus(y+I)\big) &= f\big((x+y)+I\big) \\ &= \phi(x+y) \\ &= \phi(x)+\phi(y) \\ &= f(x+I)+f(y+I)\end{aligned}$$

and

$$\begin{aligned}f\big(\lambda(x+I)\big) &= f\big((\lambda x)+I\big) \\ &= \phi(\lambda x) \\ &= \lambda\phi(x) \\ &= \lambda f(x+I)\end{aligned}$$

f also preserves the Lie bracket:

$$\begin{aligned} f\big([x+I,y+I]\big) &= f\big([x,y]+I\big) \\ &= \phi\big([x,y]\big) \\ &= \big[\phi(x),\phi(y)\big] \\ &= \big[f(x+I),f(y+I)\big] \end{aligned}$$

So f is a Lie algebra homomorphism.

Furthermore, f surjects onto the image of ϕ, since for any $\phi(x) \in \operatorname{im} f$, $f(x+I) = \phi(x)$, and f is injective since

$$\begin{aligned} \ker(f) &= \{x+I \in L/I : f(x+I) = 0\} \\ &= \{x+I \in L/I : \phi(x) = 0\} \\ &= \{x+I \in L/I : x \in \ker(\phi)\} \\ &= \{x+I \in L/I : x \in I\} \\ &= \{I\} \\ &= \{0_{L/I}\} \end{aligned}$$

So f witnesses the isomorphism $L_1/\ker(\phi) \cong \operatorname{im}(\phi)$. ■

Example. △

The other standard isomorphism theorems also hold for Lie algebras, their proofs being similar to the corresponding proofs for rings:

Theorem 1.12 (Second Isomorphism Theorem)**.** *Let L be a Lie algebra, and $I,J \subseteq L$ be ideals of L. Then,*

$$\frac{I+J}{J} \cong \frac{I}{I \cap J}$$

Theorem 1.13 (Third Isomorphism Theorem)**.** *Let L be a Lie algebra, and $I,J \subseteq L$ be ideals of L. Then, J/I is an ideal of L/I, and,*

$$\frac{L/I}{J/I} \cong L/J$$

Theorem 1.14 (Correspondence Theorem)**.** *Let L be a Lie algebra, and $I \subseteq L$ be an ideal of L. Then, there is a bijection*

$$\{J : J \subseteq I \text{ is an ideal of } L\} \cong \{K : K \text{ is an ideal of } L/I\}$$

1.7 Direct Sums

Let L_1 and L_2 be Lie algebras, and consider the cartesian product of the underlying sets:

$$L_1 \times L_2 = \big\{(x,y) : x \in L_1, y \in L_2\big\}$$

The operations on L_1 and L_2 naturally descend pointwise to this product:

$$\begin{aligned} (x,y) + (x',y') &\coloneqq (x+x', y+y') \\ \lambda(x,y) &\coloneqq (\lambda x, \lambda y) \\ \big[(x,y),(x',y')\big] &\coloneqq \big([x,y],[x',y']\big) \end{aligned}$$

Under these operations, this set is a Lie algebra, denoted by $L_1 \oplus L_2$ called the *direct sum* of L_1 and L_2.

Lemma 1.15. *Let L_1 and L_2 be Lie algebras. Then,*

(*i*) $[L_1 \oplus L_2, L_1 \oplus L_2] = [L_1, L_1] \oplus [L_2, L_2]$;

(*ii*) $Z(L_1 \oplus L_2) = Z(L_1) \oplus Z(L_2)$;

(*iii*) $\{(x,0) : x \in L_1\}$ *is an ideal of* $L_1 \oplus L_2$, *isomorphic to* L_1;

(*iv*) $\{(0,y) : y \in L_2\}$ *is an ideal of* $L_1 \oplus L_2$, *isomorphic to* L_2;

(*v*) *The projections* $\pi_i : L_1 \oplus L_2 \to L_i$ *are Lie algebra homomorphisms.*

This algebra $L_1 \oplus L_2$ is also called the *external* direct sum, since we have formed a new algebra from two unrelated algebras L_1 and L_2. In constrast, the *internal* direct sum is defined as follows:

Let $L_1, L_2 \subseteq L$ be subalgebras of a Lie algebra L such that:

(*i*) $L_1 \cap L_2 = \{0_L\}$;

(*ii*) $[L_1, L_2] = \{0_L\}$.

Then, the linear subspace $L_1 + L_2$ is naturally a Lie subalgebra of L since

$$\begin{aligned}[x+y, x'+y'] &= [x,x'] + [x,y'] + [x',y] + [y,y'] \\ &= [x,x'] + [y,y'] \\ &\in L_1 + L_2\end{aligned}$$

Lemma 1.16. *The internal direct sum* $L_1 + L_2$ *is isomorphic to the external direct sum* $L_1 \oplus L_2$.

Proof. Define the map $\phi : L_1 \oplus L_2 \to L_1 + L_2$ by $(x,y) \mapsto x + y$. Linearity is clear, and for Lie brackets, we have:

$$\begin{aligned}\phi\big([(x,y),(x',y')]\big) &= \phi\big([x,y],[x',y']\big) \\ &= [x,y] + [x',y']\end{aligned}$$

and

$$\begin{aligned}\big[\phi(x,y), \phi(x',y')\big] &= [x+y, x'+y'] \\ &= [x,x'] + [x,y'] + [x',y] + [y,y']\end{aligned}$$

Since $[L_1, L_2] = \{0_L\}$, the mixed brackets vanish and the two expressions are equal.

Then, ϕ is injective since if $\phi(x,y) = x + y = 0$, then $x = -y \in L_1 \cap L_2 = \{0_L\}$, so $x = y = 0$, and ϕ has trivial kernel. We also have that ϕ is surjective, since every element of $x + y \in L_1 + L_2$ has preimage $(x,y) \in L_1 \oplus L_2$. So ϕ is an isomorphism. ∎

2 Representations

Let L be a Lie algebra over K. A *representation* of L is a Lie algebra homomorphism

$$\phi : L \to \mathfrak{gl}(V)$$

where V is a vector space over K. If $\ker(\phi)$ is trivial, then ϕ is called *faithful*.

Example.

(*i*) Every matrix Lie algebra is "really" a faithful representation of the underlying abstract Lie algebra. For instance, the abstract Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has basis elements e,h,f and Lie brackets $[e,h] = -2e$, $[e,f] = h$, and $[f,h] = 2f$, which we often represent as matrices.

(*ii*) If L is a Lie subalgebra of $\mathfrak{gl}(V)$, then the inclusion $\iota : L \to \mathfrak{gl}(V)$ is a representation called the *natural representation* of L.

(*iii*) The zero homomorphism $\phi : L \to \mathfrak{gl}(V)$ is the *trivial representation* of L.

(*iv*) The adjoint homomorphism $\mathrm{ad} : L \to \mathfrak{gl}(L)$ is a representation called the *adjoint representation* of L. This representation is faithful if and only if $Z(L) = \{0_L\}$.

△

3 Soluble and Nilpotent Lie Algebras

3.1 Solubility

Lemma 3.1. *Let L be a Lie algebra and I an ideal of L. Then, L/I is abelian if and only if $[L,L] \subseteq I$.*

Proof. By definition, L/I is abelian if $[x + I, y + I] = 0_{L/I}$ for all $x,y \in L$, or equivalently, $[x,y] + I = 0_{L/I} = I$. This holds if and only if $[x,y] \in I$, and since $[L,L] = \big\langle [x,y] : x,y \in L \big\rangle$, this is equivalent to $[L,L] \subseteq I$. ∎

Corollary 3.1.1. *The ideal $I = [L,L]$ is the smallest ideal of L such that L/I is abelian.*

The *derived series* of L is the sequence $L^{(0)}, L^{(1)}, L^{(2)}, \ldots$, defined inductively as follows:

(*i*) $L^{(0)} = L$;

(*ii*) $L^{(k+1)} = \big[L^{(k)}, L^{(k)}\big]$.

Lemma 3.2. *For any $k \in \mathbb{N}$, $L^{(k)}$ is an ideal of L, and $L^{(0)} \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \cdots$.*

Proof. We have already seen that $[L,L]$ is an ideal of L, so the chain of containments follows by induction. The Lie bracket of ideals is also an ideal, so $L^{(k+1)} = \big[L^{(k)}, L^{(k)}\big]$ is an ideal of $L^{(0)} = L$ by induction. ∎

A Lie algebra L is *soluble* if there exists $n \in \mathbb{N}$ such that

$$L^{(n)} = \{0_L\}$$

Example.

(*i*) If L is abelian, then L is soluble. This is immediate since $L^{(1)} = [L,L] = \big\langle [x,y] : x,y \in L \big\rangle = \langle 0 \rangle = \{0\}$.

(*ii*) $L = \mathfrak{b}_n(\mathbb{C})$ is soluble for all $n \in \mathbb{N}$. The Lie bracket of any two upper tringular matrices is strictly upper triangular. Continuing to take Lie brackets, the matrices gain an additional zero diagonal at each step, and thus eventually all become the zero matrix after at most n iterations. So $L^{(n)} = \{0\}$, and $\mathfrak{b}_n(\mathbb{C})$ is soluble.

(*iii*) Let $L = \mathfrak{gl}_n(\mathbb{C})$. Since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, $[A,B] \in \mathfrak{sl}_n(\mathbb{C})$, so $L^{(1)} \subseteq \mathfrak{sl}_n(\mathbb{C})$. Conversely, $\mathfrak{sl}_n(\mathbb{C})$ is generated by the brackets $[E_{ij}, E_{jk}] = E_{ik}$ (the off-diagonal elements) and $[E_{ij}, E_{ji}] = E_{ii} - E_{jj}$ (the diagonal elements preserving zero trace), so $\mathfrak{sl}_n(C) \subseteq L^{(1)}$.

By the same argument, $[\mathfrak{sl}_n(\mathbb{C}), \mathfrak{sl}_n(\mathbb{C})] = \mathfrak{sl}_n(\mathbb{C})$, so $L^{(k)} = L^{(1)} = \mathfrak{sl}_n(\mathbb{C})$ for all $k \in \mathbb{N}$, and $\mathfrak{sl}_n(\mathbb{C}) \neq \{0\}$ for $n \geq 2$. Thus, $\mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{sl}_n(\mathbb{C})$ are not soluble for $n \geq 2$.

△

Lemma 3.3. *Let $\phi : L \to L'$ be a Lie algebra homomorphism. Then, for all $n \in \mathbb{N}$,*

$$\phi(L^{(n)}) = \phi(L)^{(n)}$$

Proof. We induct on n. For $n = 0$, $\phi(L^{(0)}) = \phi(L) = \phi(L)^{(0)}$. Now suppose the result holds for some fixed arbitrary n. Then,

$$\begin{aligned}\phi(L^{(n+1)}) &= \phi\big([L^{(n)},L^{(n)}]\big)\\ &= \big[\phi(L^{(n)}),\phi(L^{(n)})\big]\\ &= [\phi(L)^{(n)},\phi(L)^{(n)}]\\ &= \phi(L)^{(n+1)}\end{aligned}$$

■

Lemma 3.4.

(*i*) *If L is soluble, then every Lie subalgebra of L is soluble.*

(*ii*) *If L is soluble, then every homomorphic image of L is soluble.*

(*iii*) *If I is an ideal of L such that L/I and I are soluble, then L is soluble.*

(*iv*) *If I and J are soluble ideals of L, then $I + J$ is also soluble.*

Proof.

(*i*) If L is a Lie subalgebra of L, then $M^{(i)} \subseteq L^{(i)}$ for all $i \in \mathbb{N}$, so if L is soluble, there exists $n \in \mathbb{N}$ such that $L^{(n)}$ is trivial, so $M^{(n)} \subseteq L^{(n)}$ must also be trivial.

(*ii*) Let $\phi : L \to M$ be a Lie algebra homomorphism. Since $\phi(L^{(i)}) = \phi(L)^{(i)}$. Since L is soluble, $L^{(n)} = \{0_L\}$ for some $n \in \mathbb{N}$, so $\phi(L)^{(n)} = \phi(L^{(n)}) = \phi\big(\{0_L\}\big) = \{0_M\}$ is also soluble.

(*iii*) Since L/I is soluble, $(L/I)^{(n)} = \{0\}$ for some $n \in \mathbb{N}$. Let $\pi : L \to L/I$ be the natural quotient homomorphism. Then, $\{0\} = (L/I)^{(n)} = \pi(L)^{(n)} = \pi(L^{(n)})$, so $L^{(n)} \subseteq \ker(\pi) = I$. Then, I is soluble, so $I^{(m)} = \{0\}$ for some $m \in \mathbb{N}$. So $(L^{(n)})^{(m)} \subseteq \{0\}$, and L is soluble.

(*iv*) By the, second isomorphism theorem,

$$\frac{I+J}{J} \cong \frac{I}{I \cap J}$$

Since $I/(I \cap J) = \pi(I)$, it is soluble by (*ii*). So $(I + J)/J$ is also soluble, and hence $I + J$ is soluble by (*iii*).

■

3.2 Simple and Semisimple Lie Algebras

A non-abelian Lie algebra L is *simple* if it has no proper non-zero ideals. That is, the only ideals of L are $\{0\}$ and L.

Example.

(*i*) $\mathfrak{sl}_n(\mathbb{C})$ is an proper non-zero ideal of $\mathfrak{gl}_n(\mathbb{C})$ for $n \geq 2$, so $\mathfrak{gl}_n(\mathbb{C})$ is not simple for $n \geq 2$. Also, for $n = 1$, $\mathfrak{gl}_1(\mathbb{C}) \cong \mathbb{C}$ is 1-dimensional over $\mathbb{C}$ and is hence abelian (and thus also non-simple).

(*ii*) $\mathfrak{sl}_2(\mathbb{C})$ is simple.

Suppose otherwise that $\mathfrak{sl}_2(\mathbb{C})$ has a non-zero proper ideal I. Let

$$e_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

be a basis of $\mathfrak{sl}_2(\mathbb{C})$, with brackets $[e_1,e_2] = e_3$, $[e_1,e_3] = -2e_1$, and $[e_2,e_3] = 2e_2$.

Suppose $e_1 \in I$. Then, $[e_1,e_2] = e_3 \in I$, and $[e_2,e_3] = 2e_2 \in I$, so $e_2 \in I$. So $I = \mathfrak{sl}_2(\mathbb{C})$. Through similar considerations, if $e_2 \in I$, then necessarily $e_1,e_3 \in I$, and if $e_3 \in I$, then $e_1,e_2 \in I$. So any non-zero ideal must be equal to $\mathfrak{sl}_2(\mathbb{C})$, so $\mathfrak{sl}_2(\mathbb{C})$ is simple.

$\triangle$

Let L be a Lie algebra, and consider the set $\mathcal{R}$ of soluble ideals of L:

$$\mathcal{R} = \{I : I \text{ is a soluble ideal of } L\}$$

Let $R \in \mathcal{R}$ be an ideal of maximum dimension, which exists as L is finite-dimensional. For any $I \in \mathcal{R}$, $I + R$ is a soluble ideal of L, and since $R \subseteq I + R$, $\dim(R) \leq \dim(I + R)$. But the dimension of R is maximal, so $\dim(I + R) \leq \dim(R)$, so $\dim(I + R) = \dim(R)$, which holds if and only if $I \subseteq R$. So R contains every soluble ideal of L.

Moreover, R is unique, since if $R' \in \mathcal{R}$ is another soluble ideal containing every soluble ideal of L, then $R' \subseteq R$, since R' is soluble, and $R \subseteq R'$, since R is soluble.

Thus, for any Lie algebra L, there exists a unique soluble ideal of L that contains every soluble ideal of L. This ideal is denoted by $\operatorname{Rad}(L)$, and is called the *radical* of L. Since the solubility of L implies the solubility of every Lie subalgebra of L, L is soluble if and only if $\operatorname{Rad}(L) = L$.

A Lie algebra L is *semisimple* if $\operatorname{Rad}(L) = \{0_L\}$. That is, if it has no non-zero *soluble* ideals.

Example.

(*i*) $\mathfrak{sl}_2(\mathbb{C})$ is semisimple.

(*ii*) $\mathfrak{gl}_2(\mathbb{C})$ is not semisimple.

$\triangle$

Lemma 3.5. *For any Lie algebra L, $L/\operatorname{Rad}(L)$ is semisimple.*

Proof. Let K be a soluble ideal of $L/\operatorname{Rad}(L)$. By the correspondence theorem, there is a corresponding ideal I of L with $\operatorname{Rad}(L) \subseteq I$ and $I/\operatorname{Rad}(L) = K$. Since both K and $\operatorname{Rad}(L)$ are soluble, so is I. But by the definition of a radical, $I \subseteq \operatorname{Rad}(I)$, so $I = \operatorname{Rad}(L)$. So $K = \{0\}$, as required. ∎

Lemma 3.6. *If L is a complex simple Lie algebra, then L is semisimple.*

Proof. Suppose otherwise that L is not semisimple, so $\operatorname{Rad}(L) \neq \{0\}$. Since L is simple, it has no proper non-zero ideals, and hence $\operatorname{Rad}(L) = L$. So L is soluble, and $L^{(1)} = [L,L] = \{0\}$ (since L has no proper non-zero ideals, and $[L,L] = L$ contradicts solubility). But then, L is abelian, which contradicts that L is simple. ∎

3.3 Nilpotent Lie Algebras

The *lower central series* of L is the sequence $L^0,L^1,L^2,\ldots$, defined inductively as follows:

(*i*) $L^0 = L$;

(*ii*) $L^{(k+1)} = \left[L,L^k\right]$.

Lemma 3.7. *For any $k \in \mathbb{N}$, L^k is an ideal of L, and $L^0 \subseteq L^1 \subseteq L^2 \subseteq \cdots$.*

Proof. Identical to Lemma 3.2. ■

A Lie algebra L is *nilpotent* if there exists $n \in \mathbb{N}$ such that

$$L^n = \{0_L\}$$

The connection to ordinary nilpotency of operators will be made explored later.

Example.

(*i*) If L is abelian, then L is nilpotent.

(*ii*) As seen earlier, if $L = \mathfrak{sl}_n(\mathbb{C})$, $[L,L] = L$, so $L^k = L$ and $\mathfrak{sl}_n(\mathbb{C})$ is not nilpotent.

(*iii*) The Lie algebra $L = \mathfrak{u}_3(\mathbb{C})$, called the *Heisenberg* Lie algebra, is nilpotent. $L = \langle E_{12}, E_{13}, E_{23} \rangle$, and we have $[E_{12}, E_{13}] = 0$, $[E_{12}, E_{23}] = E_{13}$, $[E_{13}, E_{23}] = 0$. Then, using the structure constants above, $[A,B] = \alpha E_{13}$, so $L^1 = \langle E_{13} \rangle_{\mathbb{C}}$. Then, using the structure constants, we also have that $[A, E_{13}] = 0$ for any $A \in L$, so $L^2 = [L, L^1] = \big[L, \langle E_{13} \rangle\big] = \{0\}$.

More generally, $L = \mathfrak{u}_n(\mathbb{C})$ is nilpotent for all $n \in \mathbb{N}$.

△

Lemma 3.8.

(*i*) *If L is nilpotent, then every Lie subalgebra of L is nilpotent.*

(*ii*) *If L is nilpotent and non-trivial, then $Z(L)$ is non-trivial.*

(*iii*) *If $L/Z(L)$ is nilpotent, then L is nilpotent.*

Theorem 3.9. *Let L be a Lie algebra. Then, for all $n \in \mathbb{N}$,*

$$L^{(n)} \subseteq L^n$$

Proof. We induct on n. For $n = 0$, $L^{(0)} = L \subseteq L = L^0$. Now, suppose the inclusion holds for some arbitrary fixed n.

Since $L^{(n)} \subseteq L$, every Lie bracket $[x,y] \in [L^{(n)}, L^{(n)}]$ also lies in $[L, L^{(n)}]$, so

$$\begin{aligned} L^{(n+1)} &= [L^{(n)}, L^{(n)}] \\ &\subseteq [L, L^{(n)}] \\ &\subseteq [L, L^n] \\ &= L^{n+1} \end{aligned}$$

■

Corollary 3.9.1. *Every nilpotent Lie algebra is soluble.*

Proof. If L is nilpotent, then $L^n = \{0\}$ for some $n \in \mathbb{N}$. Then, $L^{(n)} \subseteq L^n = \{0\}$, and L is soluble. ■

3.4 Weights

Let V be a vector space over a field K. Recall that a non-zero vector $v \in V$ is an eigenvector for a linear map $T : V \to V$ if there exists an eigenvalue $\lambda \in K$ such that $T(v) = \lambda v$.

Let V be a vector space and H be a Lie subalgebra of $\mathfrak{gl}(V)$. A non-zero vector $v \in V$ is an *eigenvector* for H if v is an eigenvector for every $T \in H$.

That is, $v \in V$ is an eigenvector for H if for every $T \in H$, there exists $\lambda_T \in K$ such that $T(v) = \lambda_T v$. This induces a function $\lambda : H \to K$ that sends each transformation $T \in H$ to its eigenvalue:

$$\lambda(T) = \lambda_T$$

So equivalently, a non-zero vector $v \in V$ is an eigenvector for H if there exists a function $\lambda : H \to K$ such that $T(v) = \lambda(T)v$ for all $T \in H$.

Now, given such a function λ, consider the set V_λ of all eigenvectors of H consistent with this function:

$$V_\lambda := \{w \in V : \forall T \in H, T(w) = \lambda(T)w\}$$

By the construction of λ, we have that $v \in V$, so V is non-empty. Then, for all $T \in H$, $\alpha \in K$, and x,$y \in V_\lambda$,

$$\begin{aligned} T(x+y) &= T(x) + T(y) \\ &= \lambda(T)x + \lambda(T)y \\ &= \lambda(T)(x+y) \end{aligned}$$

so $x + y \in V_\lambda$, and

$$T(\alpha x) = \lambda(T)\alpha x$$

so $\alpha x \in V_\lambda$. So, V_λ is closed under vector addition and scaling. Also, $0 \in V_\lambda$ since the zero vector is preserved under any linear transformation T and annihilates any scalar, so V_λ is a linear subspace of V.

Moreover, for any S,$T \in H$, α,$\beta \in K$, and $w \in V_\lambda$,

$$\begin{aligned} \lambda(\alpha T + \beta S)w &= (\alpha T + \beta S)(w) \\ &= \alpha T(w) + \beta S(w) \\ &= \alpha\lambda(T)w + \beta\lambda(S)w \\ &= (\alpha\lambda(T) + \beta\lambda(T))w \end{aligned}$$

so $\lambda : H \to K$ is linear.

Let V be a vector space over a field K and H be a Lie subalgebra of $\mathfrak{gl}(V)$. A *weight* of H is a linear function $\lambda : H \to K$ such that the *weight space* V_λ is a non-trivial linear subspace of V.

Example. Consider $L = \mathfrak{gl}_3(\mathbb{C})$, and let $H = \mathfrak{b}_3(\mathbb{C}) \subseteq L$. A general element $A \in H$ has the form

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

Then, the basis vector e_1 is an eigenvector of this matrix:

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

so e_1 is an eigenvector for H. The associated weight $\lambda : H \to K$ is then given by

$$\lambda\left(\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}\right) = a$$

and the weight space V_λ is given by

$$V_\lambda = \{ae_1 : a \in K\} = \langle e_1 \rangle$$

$\triangle$

Example. Let $L = \mathfrak{sl}_3(\mathbb{C})$, and $H = \langle h_1, h_2 \rangle \subseteq L$, where

$$h_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad h_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

We have:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -y \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ -z \end{bmatrix}$$

By inspection,

- e_1 is an eigenvector for
  - h_1 with eigenvalue 1;
  - h_2 with eigenvalue 0;
- e_2 is an eigenvector for
  - h_1 with eigenvalue -1;
  - h_2 with eigenvalue 1;
- e_2 is an eigenvector for
  - h_1 with eigenvalue 0;
  - h_2 with eigenvalue -1.

So, we may define three weights for each eigenvector by mapping h_1 and h_2 to the corresponding eigenvalues:

$$\begin{aligned} \lambda_1(h_1) &= 1; & \lambda_1(h_1) &= -1; & \lambda_1(h_1) &= 0; \\ \lambda_1(h_2) &= 0; & \lambda_1(h_2) &= 1; & \lambda_1(h_2) &= -1; \\ V_{\lambda_1} &= \langle e_1 \rangle & V_{\lambda_2} &= \langle e_2 \rangle & V_{\lambda_3} &= \langle e_3 \rangle \end{aligned}$$

$\triangle$

Lemma 3.10. *Let V be a vector space over a field K of characteristic $\operatorname{char}(K) = 0$, L be a Lie subalgebra of $\mathfrak{gl}(V)$, and I be an ideal of L. Then, for any weight $\lambda : I \to K$, V_λ is L-invariant. That is, for every $T \in L$,*

$$T(V_\lambda) \subseteq V_\lambda$$

3.5 Engel's Theorem

Let L be a Lie algebra. An element $x \in L$ is *ad-nilpotent* if there exists $n \in \mathbb{N}$ such that the n-fold iteration of $\mathrm{ad}(x)$ is the zero map:

$$\mathrm{ad}(x)^n = \mathbf{0}$$

That is, $\mathrm{ad}(x)$ is a nilpotent operator in the ordinary sense.

Lemma 3.11. *If L is a nilpotent Lie algebra, then every element of L is ad-nilpotent.*

Proof. Since L is nilpotent, there exists $n \in \mathbb{N}$ such that $L^n = \{0\}$, so for every $x,y \in L$, applying the Lie bracket n times yields 0:

$$\mathrm{ad}(x)^n(y) = [\underbrace{x,[x,\cdots[x}_{n},y]\cdots]] = 0$$

■

The question is, does the converse hold?

Lemma 3.12. *Let V be a vector space, and L a Lie subalgebra of $\mathfrak{gl}(V)$. If $x \in L$ is nilpotent as a linear map, then it is ad-nilpotent.*

Proof. Since x is nilpotent, there exists n such that $x^n = \mathbf{0}$. For any $y \in L$, consider the $2n$-fold iteration of $\mathrm{ad}(x)$. By induction, we can prove that:

$$\mathrm{ad}(x)^{2n}(y) = \sum_{i=1}^{2n} \alpha_i x^i y x^{2n-i}$$

for some coefficients $\alpha_i \in K$. The factor x^{2n-i} vanishes on the first half of the sum, while x^i vanishes over the second, so the entire sum vanishes, and x is ad-nilpotent. ■

Lemma 3.13. *Let V be a vector space, and L be a Lie subalgebra of $\mathfrak{gl}(V)$ such that every element of L is nilpotent. Then, there exists a non-zero vector v such that v on all of L:*

$$\forall x \in L, x(v) = 0$$

Theorem 3.14 (Engel's Theorem for Subalgebras of $\mathfrak{gl}(V)$)**.** *Let V be a vector space, and L be a Lie subalgebra of $\mathfrak{gl}(V)$ such that every element of L is nilpotent. Then, there exists a basis of V such that the matrix of every element of L is strictly upper triangular. In particular, L is nilpotent.*

Theorem 3.15 (Engel's Theorem)**.** *Let L be a Lie algebra. If every element of L is ad-nilpotent, then L is nilpotent.*

Proof. Consider the map $\mathrm{ad} : L \to \mathrm{ad}(L) \subseteq \mathfrak{gl}(L)$. For every $x \in L$, $\mathrm{ad}(x)$ is nilpotent, so by the previous theorem, $\mathrm{ad}(L)$ is nilpotent. Then, $\mathrm{ad}(L) \cong L/\ker(\mathrm{ad}) = L/Z(L)$ is nilpotent, so L is nilpotent. ■

3.6 Lie's Theorem

Lemma 3.16. *Let $L = \mathfrak{b}_n(\mathbb{C})$. Then, $[L,L] = \mathfrak{u}_n(\mathbb{C})$.*

Theorem 3.17. *Let V be a vector space over $\mathbb{C}$ and L be a Lie subalgebra of $\mathfrak{gl}(V)$. If L is soluble, there exists an eigenvector for L. That is, there exists a non-zero $v \in V$ such that for every $x \in L$, there exists λ_x such that $x(v) = \lambda_x v$.*

Theorem 3.18 (Lie's Theorem)**.** *Let V be a vector space over $\mathbb{C}$, and L be a soluble Lie subalgebra of $\mathfrak{gl}(V)$. Then, there exists a basis of V such that the matrix of every element of L is upper triangular.*

Corollary 3.18.1. *Let V be a vector space over $\mathbb{C}$, and L be a soluble Lie subalgebra of $\mathfrak{gl}(V)$. If $x \in [L,L]$, then x is nilpotent.*

Corollary 3.18.2. *Let L be a Lie algebra over $\mathbb{C}$. Then, L is soluble if and only if $[L,L]$ is nilpotent.*

4 The Killing Form and Cartan's Criteria

4.1 Jordan Decomposition

In this section, V is a vector space over $\mathbb{C}$.

Let $x : V \to V$ be a linear map. Then, there exist linear maps $d : V \to V$ and $n : V \to V$ such that

(i) $x = d + n$;

(ii) d is diagonalisable and n is nilpotent;

(iii) $dn = nd$.

Such a decomposition $x = d + n$ is called a *Jordan decomposition* of x.

Lemma 4.1. *The Jordan decomposition of $x : V \to V$ is unique. Moreover, there exist polynomials $p,q \in \mathbb{C}[t]$ without constant terms such that $p(x) = d$ and $q(x) = n$.*

Let $\mathcal{B}$ be a basis of V such that the matrix $D = [d]_{\mathcal{B}}$ of d in $\mathcal{B}$ is diagonal. Let $\overline{d}$ be the linear map whose matrix with respect to $\mathcal{B}$ is the complex conjugate $\overline{D}$ of D. Then, there exists $\tilde{p} \in \mathbb{C}[t]$ such that $\tilde{p}(x) = \overline{d}$.

Lemma 4.2. *Let $x \in \mathfrak{gl}(V)$. If $x = d + n$ is its Jordan decomposition, then*

$$\mathrm{ad}(x) = \mathrm{ad}(d) + \mathrm{ad}(n)$$

Lemma 4.3. *For any $A,B,C \in \mathfrak{gl}(V)$,*

$$\mathrm{tr}\big([A,B]C\big) = \mathrm{tr}\big(A[B,C]\big)$$

Theorem 4.4. *Let L be a Lie subalgebra of $\mathfrak{gl}(V)$. If $\mathrm{tr}(x \circ y) = 0$ for all $x,y \in L$, then L is soluble.*

Corollary 4.4.1. *Let L be a complex Lie algebra. Then, L is soluble if and only if for all $x \in L$ and $y \in [L,L]$,*

$$\mathrm{tr}\big(\mathrm{ad}(x) \circ \mathrm{ad}(y)\big) = 0$$

4.2 The Killng Form

Let L be a complex Lie algebra. The *Killing form* on L is the map $k : L \times L \to \mathbb{C}$ defined by

$$k(x,y) \coloneqq \mathrm{tr}\big(\mathrm{ad}(x) \circ \mathrm{ad}(y)\big)$$

Lemma 4.5. *The Killing form is a symmetric bilinear form. Moreover,*

$$k\big([x,y],z\big) = k\big(x,[y,z]\big)$$

Theorem 4.6 (Cartan's First Criterion)**.** *Let L be a complex Lie algebra. Then, L is soluble if and only if $k(x,y) = 0$ for all $x \in L$ and $y \in [L,L]$.*

Lemma 4.7. *Let L be a complex Lie algebra and I be an ideal of L. Then, the restriction of the Killing form k of L to I is the Killing form k_I on I (i.e. the Killing form on I when I is considered as a complex Lie algebra itself); for all $x,y \in I$,*

$$k(x,y) = k_I(x,y)$$

Let τ be a symmetric bilinear form on a vector space V and let W be a linear subspace of V. We define the *orthogonal complement* $W^\perp$ of W in V to be:

$$W^\perp := \big\{v \in V : \forall w \in W, \tau(v,w) = 0\big\}$$

Lemma 4.8. *The set $W^\perp$ is a linear subspace of V.*

In particular, the subspace $V^\perp$ is called the *radical* of τ.

The form τ is *non-degenerate* if its radical $V^\perp = \{0_V\}$ is trivial. Note that for a symmetric bilinear form, positive-definiteness ($\tau(x,x) \neq 0$ whenever $x \neq 0$) implies non-degeneracy.

Recall that for a fixed basis $\mathcal{B} = e_1, \ldots, e_n$, a bilinear form τ is uniquely determined by the matrix $[\tau]_\mathcal{B} = \big(\tau(e_i,e_j)\big)$, and vice versa. When τ is symmetric matrix, this matrix is symmetric, and τ is non-degenerate if and only if $\det\big([\tau]_\mathcal{B}\big) \neq 0$.

A basis $\mathcal{B}$ of V is *orthonormal* if $\tau(e_i,e_i) = 1$ and $\tau(e_i,e_j)$ for all $i \neq j$.

Lemma 4.9. *For an non-degenerate symmetric bilinear form τ and a linear subspace W of V,*

$$\dim(V) = \dim(W) + \dim(W^\perp)$$

Moreover, if $W \cap W^\perp = \{0\}$, then $V = W \oplus W^\perp$, $(W^\perp)^\perp = W$, and the restrictions of τ to W and $W^\perp$ are non-degenerate.

We are interested in the orthogonal complements of ideals with respect to the Killing form.

Lemma 4.10. *Let L be a complex Lie algebra, and I an ideal of L. Then, the ortheogonal complement*

$$I^\perp = \{x \in L : \forall i \in I, k(x,i) = 0\}$$

is an ideal of L.

Proof. The orthogonal complement is always a linear subspace, so it remains to verify that $I^\perp$ absorbs Lie brackets with any element of L.

Let $x \in I^\perp$, $y \in L$, and $z \in I$. Since I is an ideal, $[y,z] \in I$, so

$$k\big([x,y],z\big) = k\big(x,[y,z]\big) = 0$$

so $[x,y]$ is orthogonal to $z \in I$, and is hence in $I^\perp$. ■

Theorem 4.11 (Cartan's Second Criterion)**.** *Let L be a complex Lie algebra. Then, L is semisimple if and only if its Killing form k is non-degenerate.*

Lemma 4.12. *Let L be a semisemple complex Lie algebra and I be an ideal of L. Then,*

(*i*) *$I \cap I^\perp = \{0\}$;*

(*ii*) *$L = I \oplus I^\perp$ as Lie algebras;*

(*iii*) *I and $I^\perp$ are semisimple as complex Lie algebras.*

Theorem 4.13. *Let L be a complex Lie algebra. Then, L is semisimple if and only if there exist simple ideals $L_1, L_2, \ldots, L_k \subseteq L$ such that $L = \bigoplus_{i=1}^k L_i$.*

4.3 Derivations

Given a field K, a K-algebra A is a vector space over K endowed with an additional bilinear multiplication operation $A \times A \to A$. Bilinearity is equivalent to multiplication distributing over addition and compatibility with scalar multiplication in the vector space.

Example. Lie algebras are K-algebras, with multiplication given by the Lie bracket. △

Let U be a K-algebra. A linear map $\delta : U \to U$ is a *derivation* if it satisfies the *Leibniz law*:

$$\delta(ab) = a\delta(b) + \delta(a)b$$

for all $a,b \in U$. That is, δ satisfies an analogue of the product rule of differentiation.

We denote by $\mathrm{Der}(U)$ the set of derivations on U.

Lemma 4.14. *The set* $\mathrm{Der}(U)$ *is a linear subspace of* $\mathrm{End}(V)$*, and in particular, is a vector space over* K*.*

Lemma 4.15. *Let L be a Lie algebra. Then,* $\mathrm{Der}(L)$ *is a LIe subalgebra of* $\mathfrak{gl}(L)$*. In particular,* $\mathrm{Der}(L)$ *is a Lie algebra with Lie bracket $[a,b] = ab - ba$.*

Example. The adjoint homomorphism is a derivation: for any $x,a,b \in L$,

$$\begin{aligned}
\mathrm{ad}(x)\big([a,b]\big) &= \big[x,[a,b]\big] \\
&= -\big[a,[b,x]\big] - \big[b,[x,a]\big] \\
&= \big[a,[x,b]\big] + \big[[x,a],b\big] \\
&= \big[a, \mathrm{ad}(x)(b)\big] + \big[\mathrm{ad}(x)(a),b\big]
\end{aligned}$$

△

For $x \in L$, we call $\mathrm{ad}(x)$ an *inner derivation* of L, and any other derivation an *outer derivation.*

Theorem 4.16 (Primary Decomposition)**.** *Let $x \in \mathfrak{gl}(V)$, and suppose the minimal polynomial of x factorises as*

$$(X - \lambda_1)^{a_1} \cdots (X - \lambda_r)^{a_r}$$

where the eigenvalues λ_i are distinct, and $a_i \geq 1$. Then, V decomposes as a direct sum of x-invariant subspaces V_i,

$$V = \bigoplus_{i=1}^{r} V_i$$

where $V_i = \ker(x - \lambda_i 1_V)^{a_i}$ is the generalised eigenspace of x with respect to λ_i.

Theorem 4.17. *Let L be a complex semisimple Lie algebra. Then, all derivations of L are inner. That is,*

$$\mathrm{ad}(L) = \mathrm{Der}(L)$$

Lemma 4.18. *Let L be a complex Lie algebra, and $\delta \in \mathrm{Der}(L)$ a derivation with Jordan decomposition $\delta = d_\delta + n_\delta$ in $\mathfrak{gl}(L)$. Then, $d_\delta,n_\delta \in \mathrm{Der}(L)$.*

Corollary 4.18.1. *Let L be a complex semisimple Lie algebra. Then, for each $x \in L$, there exists unique elements $d,n \in L$ such that:*

(i) $x = d + n$*;*

(ii) $\mathrm{ad}(d)$ *is diagonalisable and* $\mathrm{ad}(n)$ *is nilpotent;*

(iii) $[d,n] = 0$*.*

Let L be a complex semisimple Lie algebra. Then, the decomposition of an $x \in L$ into $x = d+n$ as above is called the *abstract Jordan decomposition* of x, d is the *semisimple part* of x, and n is the *nilpotent part* of x. If $n = 0$, then $x = d$ is *semisimple*, and if $d = 0$, then $x = n$ is *nilpotent*.

Note that if L is a semisimple Lie subalgebra of $\mathfrak{gl}(V)$, for V a complex vector space, then there is a potential ambiguity, in that every element of $\mathfrak{gl}(V)$ has its original Jordan decomposition as well as this abstract Jordan decomposition. However, these actually coincide:

Theorem 4.19. *Let L be a semisimple complex Lie algebra, and $\phi : L \to \mathfrak{gl}(V)$ a representation. Let $x = d+n$ be the abstract Jordan decomposition of $x \in L$. Then, the Jordan decomposition of $\phi(x) \in \mathfrak{gl}(V)$ is $\phi(x) = \phi(d) + \phi(n)$.*

5 Root Space Decompositions

5.1 Cartan Subalgebras

In this section, all the Lie algebras are complex semisimple.

Lemma 5.1. *Suppose $x_1, \ldots, x_n \in \mathfrak{gl}(V)$ are diagonalisable. Then, there exists a basis of V such that $x_1, \ldots, x_n$ are diagonal if and only if they pairwise commute.*

Let L be a Lie algebra. A *Cartan subalgebra* H is a Lie subalgebra of H such that

- (i) H is abelian.
- (ii) Every element $h \in H$ is semisimple.
- (iii) H is maximal with respect to (i) and (ii);

The existence of such a subalgebra is guaranteed, since $\{0\}$ satisfies (i) and (ii). However, the following lemma shows that we are guarantees more interesting Cartan subalgebras:

Lemma 5.2. *A semisimple complex Lie algebra L contains a non-zero Cartan subalgebra.*

At this point there is an obvious question - are Cartan subalgebras unique? The answer is no, but they do all have the same dimension.

Given $y \in L$, we define the *centraliser* of y as the set:

$$C_L(y) = \big\{x \in L : [x,y] = 0\big\}$$

More generally, the centraliser of a subset $Y \subseteq L$ is the set:

$$C_L(A) = \big\{x \in L : \forall y \in Y, [x,y] = 0\big\}$$

Lemma 5.3. *For any $y \in L$ and $Y \subseteq L$, the centralisers $C_L(y)$ and $C_L(Y)$ are Lie subalgebras of L.*

Proof. By construction, $C_L(y) = \ker\big(\mathrm{ad}(y)\big)$ and is thus a linear subspace of L. Now, suppose $a,b \in C_L(y)$. Then,

$$\begin{aligned}\big[y,[a,b]\big] &= -\big[b,[y,a]\big] - \big[a,[b,y]\big] \\ &= -[b,0] - [a,0] \\ &= 0\end{aligned}$$

so $[a,b] \in C_L(y)$. $C_L(Y)$ is closed under Lie brackets under the same reasoning, and is a linear subspace, since $C_L(Y) = \bigcap_{y \in Y} C_L(y)$ is an intersection of linear subspaces. ■

Lemma 5.4. *Let H be a Cartan subalgebra of L, and let $h_0 \in H$. Then,*

$$H \subseteq C_L(H) \subseteq C_L(h_0)$$

Proof. Since H is abelian, $H \subseteq C_L(H)$. Moreover,

$$\begin{aligned} C_L(H) &= \bigcap_{h \in H} C_L(h) \\ &\subseteq C_L(h_0) \end{aligned}$$

■

Lemma 5.5. *Let H be a Cartan subalgebra of L, and $h_0 \in H$ satisfying*

$$\dim C_L(h_0) \leq \dim C_L(h)$$

for all $h \in H$. Then, $C_L(h_0) = C_L(H)$.

5.2 Dual Spaces

Given a vector space V over a field K, the *dual space* V^* of V is the set of all linear functionals $V \to K$ equipped with the natural vector space structure of addition and scalar multiplication of linear maps.

Example. The weights of a subalgeba M are elements of M^*. △

Given a basis $e_1, \dots, e_n$ of V, the *dual basis* $f_1, \dots, f_n : V \to K$ of V^* is defined as:

$$f_i(e_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

For each $v \in V$, the *evaluation map* $\epsilon_v : V^* \to K$ is defined by

$$\epsilon_v(f) = f(v)$$

This map is linear and thus belongs to V^{**}. The map $\epsilon : V \to V^{**} : v \mapsto \epsilon_v$ is an isomorphism and shows that $V \cong V^{**}$.

Given a bilinear form $\tau : V \times V \to K$, we may define a linear map $\Phi_\tau : V \to V^*$ by $\phi(v) = \tau(v,-)$, with linearity of Φ_τ following from the bilinearity of τ. Conversely, given a linear map $\Phi : V \to V^*$, we may define a bilinear form $\tau_\Phi : V \times V \to K$ by $\tau_\Phi(u,v) = \Phi(u)(v)$. These operations are inverse, in that $\Phi = \Phi_{\tau_\Phi}$ and $\tau = \tau_{\Phi_\tau}$. Finally, if Φ is an isomorphism, then it has trivial kernel and τ_Φ is non-degenerate, and vice versa; if τ is non-degenerate, then Φ_τ is an isomorphism.

5.3 Roots of L Relative to a Cartan Subalgebra H

Let L be a complex semisimple Lie algebra, and suppose that H is a Cartan subalgebra of L. Then, H act on L via the adjoint map

$$\mathrm{ad}(h) : L \to L$$

for each $h \in H$.

Since H is abelian and consists of semisimple elements (i.e. $\mathrm{ad}(h)$ is diagonalisable for all $h \in H$), there exists a basis of L consisting of common eigenvectors for all elements of H (rather, for the elements $\mathrm{ad}(h)$, so this is an abuse of notation) If v is such a common eigenvector, then for each $h \in H$, there exists $\alpha(h) \in \mathbb{C}$ such that

$$\mathrm{ad}(h)(v) = \alpha(h)v$$

By definition, $\alpha : H \to \mathbb{C}$ is a weight of H (really of $\mathrm{ad}(H) \cong H$, with the isomorphism coming from the fact that L is semisimple), and so $\alpha \in H^*$. Let L_α be the corresponding weight space of H, which is

$$L_\alpha = \{x \in L : \forall h \in H, [h,x] = \alpha(h)x\} \neq \{0\}$$

In particular, if $\alpha = 0$,

$$L_0 = \{x \in L : \forall h \in H, [h,x] = 0\} = C_L(H)$$

Let $\Phi = \{\alpha \in H^* : \alpha \neq 0, L_\alpha \neq \{0\}\}$. Then, Φ is a *set of roots* of L *relative to* H, and for each $\alpha \in \Phi$, the corresponding *root space* is L_α.

By the primary decomposition theorem,

$$L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

Note that $|\Phi|$ is finite, since L is finite dimensional.

Lemma 5.6. *Let $\alpha,\beta \in H^*$. Then,*

(*i*) $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$;

(*ii*) *If $\alpha + \beta \neq 0$, then $k(L,\alpha,L_\beta) = 0$;*

(*iii*) $L_0 \cap L_0^\perp = \{0\}$, *and so $k\big|_{L_0}$ is non-degenerate.*

Theorem 5.7. *Let L be a complex semisimple Lie algebra and H be a Cartan subalgebra of L. Then, $H = C_L(H)$.*

Corollary 5.7.1. *The root space decomposition of L relative H is*

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

5.4 Sets of Roots Relative to H

Let L be a complex semisimple Lie algebra and H be a Cartan subalgebra of L.

Lemma 5.8. *For each non-zero $h \in H$, there exists $\alpha \in \Phi$ with $\alpha(h) \neq 0$.*

Corollary 5.8.1. *The set of roots span the dual space H^*.*

Proof. Suppose otherwise. Then, there exists a non-zero $h \in H$ such that $\alpha(h) = 0$ for all $\alpha \in \Phi$, contradicting the previous lemma. ∎

Lemma 5.9. *If $\alpha \in \Phi$, then $-\alpha \in \Phi$.*

Proof. Suppose otherwise there exists $\alpha \in \Phi$ such that $-\alpha \notin \Phi$. Then, for each $\beta \in \Phi \cup \{0\}$, $k(L_\alpha, L_\beta) = 0$ by Lemma 5.6. Thus $L_\alpha \subseteq L^\perp = \{0\}$, so $L_\alpha = \{0\}$, which is a contradiction. ∎

Lemma 5.10. *For each $\alpha \in \Phi$, there exists a non-zero $t_\alpha \in H$ such that for all $x \in L_\alpha$ and $y \in L_{-\alpha}$,*

$$[x,y] = k(x,y)t_\alpha$$

Moreover, $k(t_\alpha, h) = \alpha(h)$ for all $h \in H$.

Corollary 5.10.1. *If $\alpha \in \Phi$, then $L_\alpha, L_{-\alpha} = \langle t_\alpha \rangle_{\mathbb{C}}$.*

Fix a root $\alpha \in \Phi$ and fix non-zero $x \in L_\alpha$ and non-zero $y \in L_{-\alpha}$ such that $[x,y] \neq 0$. We define the set

$$M_\alpha = \big\langle x,y,[x,y] \big\rangle_{\mathbb{C}}$$

The previous two results show that $[x,y] = \lambda t_\alpha$, where $\lambda = k(x,y) \neq 0$.

Lemma 5.11. *M_α is a Lie subalgebra of L of dimension* $\dim(M_\alpha) = 3$.

Lemma 5.12. *For $\alpha \in \Phi$, $\alpha(t_\alpha) \neq 0$.*

Since $\alpha(t_\alpha) \neq 0$, we have that $k(t_\alpha,t_\alpha) = \alpha(t_\alpha) \neq 0$, so we define:

$$e_\alpha \coloneqq x, \qquad h_\alpha \coloneqq \frac{2}{k(t_\alpha,t_\alpha)} t_\alpha, \qquad f_\alpha = \frac{2}{k(t_\alpha,t_\alpha)k(x,y)} y$$

so

$$M_\alpha = \langle e_\alpha,h_\alpha,f_\alpha \rangle_{\mathbb{C}}$$

Lemma 5.13. *For every root $\alpha \in \Phi$, $M_\alpha \cong \mathfrak{sl}_2(\mathbb{C})$.*

Lemma 5.14. *For $\alpha \in \Phi$, $t_\alpha = -t_{-\alpha}$, $h_\alpha = -h_{-\alpha}$, and $\alpha(h_\alpha) = 2$.*

6 Representations

6.1 Modules

Let L be a Lie algebra over K. An *L-module* is a vector space V over K equipped with a bilinear map

$$\cdot : L \times V \to V$$

compatible with the Lie bracket in that

$$[x,y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$$

Such a map is said to be an *action* of L on V. We often drop the $\cdot$ and write the action as multiplication.

An L-module V is *trivial* if $x \cdot v = 0$ for all $x \in L$ and $v \in V$.

If V is a vector space and L is a Lie subalgebra of $\mathfrak{gl}(V)$, then the map $x \cdot v \coloneqq x(v)$ defines an action of L on V.

Example. Let $V = \mathbb{C}^3$ and $L = \mathfrak{b}_3(\mathbb{C}) \subseteq \mathfrak{gl}(\mathbb{C}^3)$. We define a bilinear map $L \times V \to V$ by $A \cdot v = Av$. Then,

$$\begin{aligned} [A,B] \cdot v &= (AB - BA)v \\ &= ABv - BAv \\ &= A(Bv) - B(Av) \\ &= A \cdot (B \cdot v) - B \cdot (A \cdot v) \end{aligned}$$

so this map is an action of L on V. △

Lemma 6.1. *Let L be a Lie algebra and $\phi : L \to \mathfrak{gl}(V)$ be a representation of L. Then, V is an L-module under the action defined by $x \cdot v \coloneqq \phi(x)(v)$.*

Conversely, if V is an L-module, then there is a corresponding representation $\phi : L \to \mathfrak{gl}(V)$ defined by $\phi(x)(v) \coloneqq x \cdot v$.

Let L be a Lie algebra and V be an L-module. An *L-submodule* of V is a linear subspace W of V which is also an L-module under the same action $x \cdot v$ as for V.

To check that W is a submodule of an L-module V, it suffices to check that W is L-invariant.

Example. Let $V = \mathbb{C}^3$ and $L = \mathfrak{b}_3(\mathbb{C}) \subseteq \mathfrak{gl}(\mathbb{C}^3)$ as above, and consider the linear subspace $W = \langle e_1 \rangle \subseteq V$. Then, for any $A \in L$ and $v \in W$,

$$Av = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} ax \\ 0 \\ 0 \end{bmatrix} \in W$$

so W is an L-submodule of V. △

Example. Let V and L be as above, and let $U = \langle e_1,e_2 \rangle \subseteq V$. Then, for any $A \in L$ and $v \in U$,

$$Av = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} ax+by \\ dy \\ 0 \end{bmatrix} \in U$$

so U is also an L-submodule of V. △

Let L be a Lie algebra and V,W be L-modules. An *L-module homomorphism* from V to W is a linear map $\phi : V \to W$ such that

$$x \cdot \phi(v) = \phi(x \cdot v)$$

for all $x \in L$ and $v \in V$. If ϕ is further an isomorphism of vector spaces, then it is an *L-module isomorphism.* As usual, we say that V and W are isomorphic if there exists an L-module isomorphism between them, and we write $V \cong W$ to denote for this relation.

Let L be a Lie algebra and V an L-module. Then, V is the *direct sum* of U and W if $V = U \oplus W$ as vector spaces, and both U and W are L-submodules of V.

One may verify that if we define an external direct sum of two L-modules U and W by the action $x \cdot (u + w) = x \cdot u + x \cdot w$ on the vector space $U \oplus W$, we obtain an L-module isomorphic to the internal direct sum.

A L-module V is *irreducible* or *simple* if V is non-trivial and has no proper non-zero submodules. That is, the only submodules of V are $\{0\}$ and V.

An L-module V is *completely reducible* if for any L-submodule W of V, there exists an L-submodule W' of V such that $V = W \oplus W'$.

A module V is *indecomposable* if it cannot be expressed as the direct sum of two non-zero proper L-submodules of V.

Lemma 6.2. *For any L-module V, irreducibility implies indecomposability, but not the converse in general.*

Theorem 6.3 (Weyl's Theorem)**.** *A non-zero module of a semisimple complex Lie algebra is completely reducible.*

6.2 Representation Theory of $\mathfrak{sl}_2(\mathbb{C})$

In this section, we classify the irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules. As usual, let $e = E_{12}$,$h = E_{11} - E_{22}$,$f = E_{21}$ be the standard basis of $\mathfrak{sl}_2(\mathbb{C})$.

Consider the vector space $\mathbb{C}[X,Y]$ of polynomials in two indeterminates X and Y. For each $d \geq 0$, define W_d to be the linear subspace of homogeneous degree-d polynomials in X and Y. A basis for W_d is then given by X^d,$X^{d-1}Y, \ldots XY^{d-1}$,$Y^d$, so $\dim(W_d) = d + 1$.

We define an action of $\mathfrak{sl}_2(\mathbb{C})$ on W_d as to make W_d into an $\mathfrak{sl}_2(\mathbb{C})$-module. To do this, it is sufficient to define the action of e, h, and f on $p \in W_d$:

$$e \cdot p = X \frac{\partial p}{\partial Y}, \qquad h \cdot p = X \frac{\partial p}{\partial X} - Y \frac{\partial p}{\partial Y}, \qquad f \cdot p = Y \frac{\partial p}{\partial X}$$

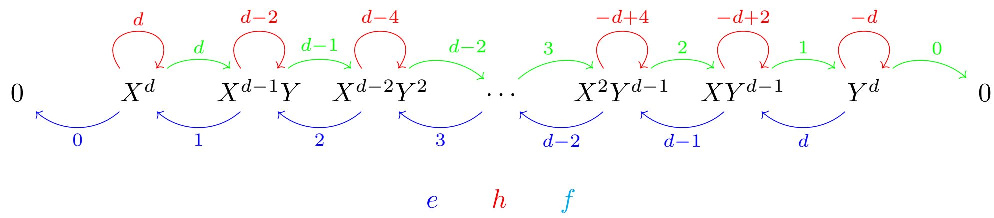
So, on monomials, the actions are:

$$e \cdot X^aY^b = bX^{a+1}Y^{b-1}, \qquad h \cdot X^aY^b = (a-b)X^aY^b, \qquad f \cdot X^aY^b = bX^{a-1}Y^{b+1}$$

Theorem 6.4. *W_d is an $\mathfrak{sl}_2(\mathbb{C})$ module with this action.*

Lemma 6.5. *For any two basis vectors $v_1 = X^aY^{d-a}$ and $v_2 = X^bY^{d-b}$, there exists a sequence of elements $\ell_1, \ldots, \ell_n \in \mathfrak{sl}_2(\mathbb{C})$ such that $\ell_1 \cdot (\ell_2 \cdot (\cdots (\ell_n \cdot v_1))) = v_2$.*

Proof. It is sufficient to show that we may reach, starting from X, to $X^{d-1}Y$, to ... to Y^d, and vice versa. $f \cdot X = dX^{d-1}Y$, so the element $\frac{1}{d}f$ maps X^d to $X^{d-1}Y$. In general, to obtain $X^{d-a-1}Y^{a+1}$ from $X^{d-a}Y^a$, $\frac{1}{d-a}f$ will do. Similarly, applying scaled copies of e moves from Y^d to X^d:



∎

Theorem 6.6. *Each W_d for $d \geq 0$ is irreducible.*

We now want to show that any other irreducible $\mathfrak{sl}_2(\mathbb{C})$-module is isomorphic to W_d for some $d \geq 0$. This completes the classification, since W_d is isomorphic to $W_{d'}$ if and only if $d = d'$, since they have different dimensions otherwise.

Lemma 6.7. *Let V be an $\mathfrak{sl}_2(\mathbb{C})$-module, and let $v \in V$ be an eigenvalue of h with eigenvalue λ. Then,*

(i) Either $h \cdot (e^n \cdot v) = (\lambda + 2n)(e^n \cdot v)$ or $e^n \cdot v = 0$;

(ii) Either $h \cdot (f^n \cdot v) = (\lambda - 2n)(f^n \cdot v)$ or $f^n \cdot v = 0$.

We write $x^n \cdot v$ for

$$x^n \cdot v := \underbrace{x \cdot (x \cdot (\cdots (x}_{n} \cdot v) \cdots))$$

Lemma 6.8. *Let V be an $\mathfrak{sl}_2(\mathbb{C})$-module. Then, V contains an eigenvector w for h such that $e \cdot w = 0$ and $f^d \cdot w \neq 0$ but $f^{d+1} \cdot w = 0$ for some $d \geq 0$.*

Theorem 6.9. *Let V be an irreducible $\mathfrak{sl}_2(\mathbb{C})$-module. Then, V is isomorphic to W_d for some $d \geq 0$.*

6.3 The Importance of $\mathfrak{sl}_2(\mathbb{C})$ for Semisimple Complex Lie Algebras

In this section, let L be a complex semisimple Lie algebra, H be a Cartan subalgebra of L, and Φ be the set of roots relative to H. Recall that we have the decomposition of L:

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

We also defined $M_\alpha = \langle e_\alpha, h_\alpha, f_\alpha \rangle$, a subalgebra of L isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ for each root $\alpha \in \Phi$. L can be viewed as an L-module using the adjoint representation $x \cdot y = [x,y]$, and since M_α is a subalgebra of L, me may study L as an M_α-module by restricting ad_L to M_α (so the action is the same for $x \in M_\alpha$, $y \in L$).

Lemma 6.10. *If V is an M_α-submodule of L, then the eigenvalues of h_α acting on V are integers.*

We have already seen that $-\alpha \in \Phi$ if $\alpha \in \Phi$. It turns out that these are the only multiples of α in Φ:

Theorem 6.11. *For each $\alpha \in \Phi$, if $c\alpha \in \Phi$ for some $c \in \mathbb{C}$, then $c = \pm 1$.*

We define the M_α-submodules of L

$$U_\alpha \coloneqq \langle H, L_{c\alpha} \mid c\alpha \in \Phi \rangle_{\mathbb{C}} \subseteq L$$

and

$$K_\alpha \coloneqq \ker(\alpha) \subseteq L$$

Corollary 6.11.1. *For any $\alpha \in \Phi$, $U_\alpha = K_\alpha \oplus M_\alpha$.*

Corollary 6.11.2. *For any $\alpha \in \Phi$,* $\dim(L_\alpha) = 1$.

If $\beta \in \Phi \cup \{0\}$, the *$\alpha$-root string through β* is the space

$$S \coloneqq \bigoplus_c L_{\beta + c\alpha}$$

where the sum is taken over all $c \in \mathbb{Z}$ such that $\beta + c\alpha \in \Phi$. Since $[L_\gamma, L_\delta] \subseteq L_{\gamma+\delta}$ for any roots $\gamma, \delta \in \Phi$, it follows that S is an M_α-submodule of L.

Strictly speaking, the proper definition of a root string should have the sum range over all $c \in \mathbb{C}$ such that $\beta + c\alpha \in \Phi$, but it turns out that these give the same submodule of L, and we will only need to work with the above definition.

Lemma 6.12. *Let $\alpha, \beta \in \Phi$ such that $\alpha \neq \beta$. Then,*

(i) $\beta(h_\alpha) \in \mathbb{Z}$;

(ii) *There exist integers $q, r \geq 0$ such that, given an integer $k \in \mathbb{Z}$, $\beta + k\alpha \in \Phi$ if and only if $-r \leq k \leq q$. Moreover,*

$$r - q = \beta(h_\alpha)$$

(iii) $\beta - \beta(h_\alpha)\alpha \in \Phi$.

Lemma 6.13. *For $\alpha, \beta \in \Phi$, $k(h_\alpha, h_\beta) \in \mathbb{Z}$, and $k(t_\alpha, t_\beta) \in \mathbb{Q}$.*

7 Root Systems and Classifications

7.1 Roots of L

Recall that we have an explicit isomorphism between H and H^*, given by

$$h \mapsto k(h, -)$$

Furthermore, for every $\alpha \in \Phi$, there exists $t_\alpha \in H$ such that $\alpha(-) = k(t_\alpha, -)$. Now, let $\phi \in H^*$, and denote by t_ϕ the element of H satisfying

$$t_\phi \mapsto k(t_\phi, -) = \phi(-)$$

We define a bilinear form $(-,-) : H^* \times H^* \to \mathbb{C}$ by

$$(\theta, \phi) = k(t_\theta, t_\phi)$$

Since k is a symmetric bilinear form on H, $(-,-)$ is a symmetric bilinear form on H^*. In particular, for $\alpha, \beta \in \Phi$, $(\alpha, \beta) = k(t_\alpha, t_\beta) \in \mathbb{Q}$.

Since $H^* = \langle\Phi\rangle_{\mathbb{C}}$, there exist $\alpha_1, \ldots, \alpha_\ell \in \Phi$ that form a basis of H^*. We define the real subspace E of H^* by

$$E \coloneqq \mathbb{R}[\alpha_1, \ldots, \alpha_\ell]$$

Clearly, $\phi \subseteq E$, and we may restrict $(-,-)$ to E, so

$$(-,-) : E \times E \to \mathbb{R}$$

is also a symmetric bilinear form. Then, there exists $t_\theta \in H$ such that

$$\begin{aligned}
(\theta,\theta) &= k(t_\theta,t_\theta) \\
&= \operatorname{tr}\big(\operatorname{ad}(t_\theta)^2\big) \\
&= \sum_{\gamma\in\Phi} \gamma(t_\theta)^2 \\
&= \sum_{\gamma\in\Phi} k(t_\gamma,t_\theta)^2 \\
&= \sum_{\gamma\in\Phi} (\gamma,\theta)^2
\end{aligned}$$

Since $(\gamma,\theta) \in \mathbb{Q} \subseteq \mathbb{R}$, and $(\theta,\theta) \geq 0$, $(\theta,\theta) = 0$ if and only if $(\gamma,\theta) = \gamma(t_\theta) = 0$ for all $\gamma \in \Phi$, which means that $\theta = 0$. So, E is in fact an inner product space, and in particular, a Euclidean space, since it is finite-dimensional and real-valued.

Lemma 7.1. *Let L be a semisimple complex Lie algebra with roots Φ. Then,*

(i) E is a vector space over R with a real-valued inner product;

(ii) $\langle\Phi\rangle_{\mathbb{R}} = E$, and $0 \notin \Phi$;

(iii) If $\alpha \in \Phi$, then $-\alpha \in \Phi$;

(iv) If $r\alpha \in \Phi$ for some $r \in \mathbb{R}$, then $r = \pm 1$;

(v) For $\alpha,\beta \in \Phi$,

$$\begin{aligned}
2\frac{(\beta,\alpha)}{(\alpha,\alpha)} &= k\left(t_\beta, \frac{2}{k(t_\alpha,t_\alpha)}t_\alpha\right) \\
&= k(t_\beta,h_\alpha) \\
&= \beta(h_\alpha) \\
&\in \mathbb{Z}
\end{aligned}$$

and

$$\beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$$

7.2 Root Systems

Let E be a finite-dimensional vector space over $\mathbb{R}$, and let $(-,-) : E \times E \to \mathbb{R}$ be an inner product.

For every non-zero vector $v \in E$, we define the map $\sigma_v : E \to E$ by

$$\sigma_v(x) = x - 2\frac{(x,v)}{(v,v)}v$$

Geometrically, this is the reflection through the hyperplane orthogonal to v, since $\sigma_v(v) = v - 2v = -v$ and if x is orthogonal to v, then $\sigma_v(x) = x - 0v = x$.

For $u,v \in E$, we abbreviate

$$\langle x,v \rangle \coloneqq 2\frac{(x,v)}{(v,v)}$$

Geometrically, $\langle \alpha,\beta \rangle = 2\frac{\|\alpha\|}{\|\beta\|}\cos(\theta)$, where θ is the angle between α and β, and can thus be interpreted as a normalised/rescaled projection of α onto β.

Also note that this mapping is linear in the first argument, but *not* the second.

Lemma 7.2. *For $x,y,v \in E$, $\big(\sigma_v(x),\sigma_v(y)\big) = (x,y)$.*

A subset $R \subseteq E$ is a *root system* in E if:

(R1) R is finite, $0 \notin R$, and $\langle R \rangle_{\mathbb{R}} = E$;

(R2) If $\alpha \in R$, then $c\alpha \in R$ if and only if $c = \pm 1$;

(R3) If $\alpha \in R$, then $\sigma_\alpha(R) \subseteq R$ (that is, R is closed under reflections through roots);

(R4) If $\alpha,\beta \in R$, then $\langle \alpha,\beta \rangle = 2\frac{(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$;

Let L be a semisimple Lie algebra over $\mathbb{C}$ with H a Cartan subalgebra of L, and let Φ be the set of roots of L relative to H. As before, let $E = \mathbb{R}[\Phi]$, which is a real vector space with inner product induced by the Killing form k.

For the rest of this section, let R be a root system in E.

Lemma 7.3. *For $\alpha,\beta \in R$ with $\alpha \neq \pm\beta$,*

$$\langle \alpha,\beta \rangle \langle \beta,\alpha \rangle \in \{0,1,2,3\}$$

Corollary 7.3.1. *Let $\alpha,\beta \in R$. Then,*

(i) $\langle \alpha,\beta \rangle = 0$ *if and only if a and b are orthogonal;*

(ii) $\langle \alpha,\beta \rangle > 0$ *if and only if* $\langle \beta,\alpha \rangle > 0$.

Let $\alpha,\beta \in R$, and without loss of generality, suppose $(\beta,\beta) \geq (\alpha,\alpha)$. Then,

$$\big|\langle \beta,\alpha \rangle\big| = 2\frac{\big|(\beta,\alpha)\big|}{(\alpha,\alpha)} \geq 2\frac{\big|(\beta,\alpha)\big|}{(\beta,\beta)} = \big|\langle \alpha,\beta \rangle\big|$$

We can now use the previous lemma to classify all the possible values of $\langle \beta,\alpha \rangle$:

$\langle \alpha,\beta \rangle$	$\langle \beta,\alpha \rangle$	θ	$\frac{(\beta,\beta)}{(\alpha,\alpha)} = \frac{\|\beta\|^2}{\|\alpha\|^2}$
0	0	$\frac{\pi}{2}$	undefined
1	1	$\frac{\pi}{3}$	1
-1	-1	$\frac{2\pi}{3}$	1
1	2	$\frac{\pi}{4}$	2
-1	-2	$\frac{3\pi}{4}$	2
1	3	$\frac{\pi}{6}$	3
-1	-3	$\frac{5\pi}{6}$	3

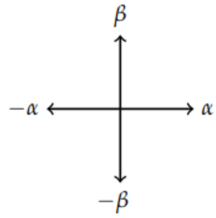
Lemma 7.4. *Let $\alpha,\beta \in R$ be such that $(\beta,\beta) \geq (\alpha,\alpha)$, and let θ be the angle between α and β. Then,*

(i) If $\frac{\pi}{2} < \theta < \pi$, then $\alpha + \beta \in R$;

(ii) If $0 < \theta < \frac{\pi}{2}$, *then* $\alpha - \beta \in R$.

Proof. $\sigma_\beta(\alpha) = \alpha - \langle\alpha,\beta\rangle\beta \in R$. If $\frac{\pi}{2} < \theta < \pi$, then from the table above, $\langle\alpha,\beta\rangle = -1$, so $\sigma_\beta(\alpha) = \alpha + \beta \in R$. Similarly, if $0 < \theta < \frac{\pi}{2}$, then $\langle\alpha,\beta\rangle = 1$, and $\sigma_\beta(\alpha) = \alpha - \beta \in R$. ■

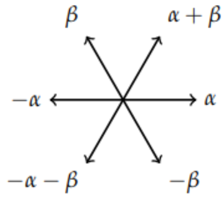
(i) $\theta = \frac{\pi}{2}$: $\alpha \perp \beta$ and $R = \{\pm\alpha, \pm\beta\}$ is a root system, which we say is of type $A_1 \times A_1$.



(ii) $\theta = \frac{2\pi}{3}$: Looking at Table 1 we have $||\alpha|| = ||\beta||$ and $\alpha + \beta \in R$ by Lemma 6.2.4. Then

$$R = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$$

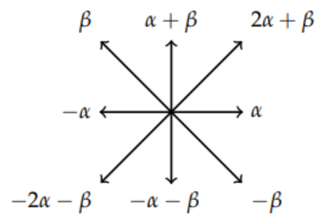
is a root system, which we say is of type A_2.



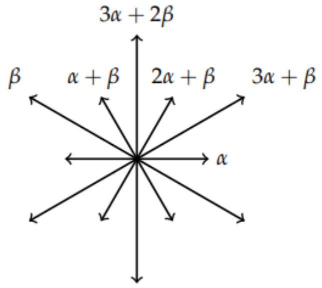
(iii) $\theta = \frac{3\pi}{4}$: Looking at Table 1 we see that $\frac{||\beta||}{||\alpha||} = \sqrt{2}$ and $\alpha + \beta \in R$ by Lemma 6.2.4. Also

$$\sigma_\alpha(\beta) = \beta - \langle\beta, \alpha\rangle\alpha = \beta - (-2)\alpha = \beta + 2\alpha \in R.$$

We claim that $R = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(\beta + 2\alpha)\}$ is a root system, which we say is of type B_2.



(iv) $\theta = \frac{5\pi}{6}$: this gives the root system of type G_2. The roots are $R = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta), \pm(3\alpha+\beta), \pm(3\alpha+2\beta)\}$.



(We have not labelled the negative roots to keep the diagram neat.)

A root system R is *irreducible* if R cannot be expressed as a disjoint union of two root systems R_1 and R_2 satisfying $(r_1,r_2) = 0$ for $r_1 \in R_1, r_2 \in R_2$.

Example.

(*i*) $A_1 \times A_1$ is not irreducible since it is the union of two root system of type A_1.

(*ii*) The root systems of type A_2, B_2, and G_2 are all irreducible. Indeed, the only 1-dimensional root system is of type A_1, so the only reducible root system in $\mathbb{R}^2$ is $A_1 \times A_1$.

$\triangle$

Lemma 7.5. *Let R be a root system in E. Then, there exist non-empty subsets $R_1,\ldots,R_\ell$ of R such that*

(*i*) $R = \bigsqcup_{i=1}^{\ell} R_i$;

(*ii*) *R_i is an irreducible root system in $E_i = \langle R_i\rangle_{\mathbb{R}}$;*

(*iii*) *$E = \bigoplus_{i=1}^{\ell} E_i$, with E_i and E_j orthogonal for $1 \le i \ne j \le \ell$.*

Let R and R' be root systems of E and E' respectively. Then, R and R' are *isomorphic* if there exists an isomorphict $\phi : E \to E'$ such that

(*i*) $\phi(R) = R'$;

(*ii*) For all $\alpha,\beta \in R$, $\langle\phi(\alpha),\phi(\beta)\rangle = \langle\alpha,\beta\rangle$.

7.3 Bases of Root Systems

Let R be a root system. A subset $\mathcal{B} \subseteq R$ is a *base* of R if:

(B1) $\mathcal{B}$ is a basis for E;

(B2) We may express any β as a $\mathbb{Z}$-linear combination of element of $\mathcal{B}$:

$$\beta = \sum_{\alpha \in \mathcal{B}} c_\alpha \alpha$$

where $c_\alpha \in \mathbb{Z}$ for all $\alpha \in \mathcal{B}$. Moreover, either the coefficients c_α are all non-negative, or all non-positive.

The roots in a base $\mathcal{B}$ are then called *simple*.

Given (B1), the expression of β in (B2) is unique.

Let β be a root, and $\beta = \sum_{\alpha \in \mathcal{B}} c_\alpha \alpha$ be the unique expression of β in terms of the base $\mathcal{B}$. Then, the *height* of β is the sum of the coefficients of the expression:

$$\sum_{\alpha \in \mathcal{B}} c_\alpha$$

If the height of β is positive, then we say that β is a *positive root*, and similarly, if the height of β is negative, then β is a *negative root*. We denote by R^+ and R^- the sets of positive and negative roots, respectively.

Note that a root cannot be simultaneously positive and negative, since that would require that every coefficient c_α is zero, in which case the root is zero, which is disallowed in the definition of a root system. So $R^+ \cap R^- = \varnothing$.

Lemma 7.6. *Let α_1,$\alpha_2 \in \mathcal{B}$ be distinct simple roots. Then, the angle between them is at least $\frac{\pi}{2}$.*

Proof. Suppose otherwise. Then, $\alpha_1 - \alpha_2 \in R$. This is a $\mathbb{Z}$-linear combination with both positive and negative coefficients, contradicting (B2). ∎

To find a base, we can pick any hyperplane in $E = \mathbb{R}^n$ which does not contain any roots. Then, label one side of the hyperplane as positive, and the other as negative. Then, the n nearest roots to the hyperplane form a base.

Theorem 7.7. *Every root system has a base.*

7.4 The Weyl Group of a Root System

By the definition of a root system, for each root $\alpha \in R$, the corresponding reflection σ_α is an element of $GL(E)$, the group of invertible linear transformations on E.

The *Weyl group* of a root system R is the group

$$W = W(R) \coloneqq \langle \sigma_\alpha \mid \alpha \in R \rangle \leq GL(E)$$

For each $\alpha \in R$, we have that $\sigma_\alpha(R) \subseteq R$, and since $\subseteq_\alpha$ is a reflection, it is an automorphism of E and thus $\sigma_\alpha(R)$ is a permutation of the roots in R. So, there exists a group homomorphism

$$f : W \to \mathrm{Sym}_{|R|}$$

sending each $w \in W$ to its action on R, viewed as a permutation.

Lemma 7.8. *The Weyl group is a subgroup of* $\mathrm{Sym}_{|R|}$. *In particular, W is finite.*

Proof. It suffices to show that f is injective. If $w \in \ker(f)$, then $f(w) = \mathrm{id}$, so w must have been the identity on R. But since R spans E, w is also the identity on E. So $\ker(f)$ is trivial, and f is injective. ∎

Lemma 7.9. *If $\alpha \in \mathcal{B}$, then the reflection σ_α permutes the set of positive roots apart from α.*

Proof. Suppose $\beta \in R^+$, and $\alpha \neq \beta$. Then,

$$\beta = \sum_{\gamma \in \mathcal{B}} c_\gamma \gamma$$

with every c_γ non-negative. Since $\alpha \neq \beta$, there must exist $\gamma \in \mathcal{B} \setminus \{\alpha\}$ such that $\alpha_\gamma > 0$ (otherwise, β is a positive multiple of α, and $\pm\alpha$ are the only multiples of α in R).

Now, $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$, so the expansion of $\sigma_\alpha(\beta)$ differs from the expansion of β only in c_α (by precisely $-\langle \beta, \alpha \rangle$). In particular, c_γ is still positive, so every coefficient is still positive, and $\sigma_\alpha(\beta) \in R^+$. ■

Lemma 7.10. *Given $\alpha \in R$, there exists $g \in W_0 \coloneqq \langle \sigma_\alpha \mid \alpha \in \mathcal{B} \rangle$ and $\beta \in \mathcal{B}$ such that $\alpha = g(\beta)$.*

Lemma 7.11. *Suppose $\alpha \in R$ and $g \in W$. Then,*

$$g \sigma_\alpha g^{-1} = \sigma_{g(\alpha)}$$

Theorem 7.12. *Let R be a root system, $\mathcal{B}$ be a base of R, and W be its Weyl group. Then,*

(i) $W = \langle \sigma_\alpha \mid \alpha \in \mathcal{B} \rangle$;

(ii) For each $\alpha \in R$, there exist $w \in W$ and $\alpha_i \in \mathcal{B}$ such that $w(\alpha_i) = \alpha$.

(iii) If $\mathcal{B}'$ is another base of R, then there exists $g \in W$ such that $\mathcal{B}' = \big\{ g(\alpha_i) : \alpha_i \in \mathcal{B} \big\}$.

7.5 Cartan Matrices and Dynkin Diagrams

Let R be a root system in E with a base $\mathcal{B} = \{\alpha_1, \ldots, \alpha_\ell\}$. The *Cartan matrix* of R is the $\ell \times \ell$ matrix

$$\big(\langle \alpha_i, \alpha_j \rangle \big)_{1 \le i,j \le \ell} = \begin{bmatrix} 2 & \langle \alpha_1, \alpha_2 \rangle & \langle \alpha_1, \alpha_3 \rangle & \cdots & \langle \alpha_1, \alpha_\ell \rangle \\ \langle \alpha_2, \alpha_1 \rangle & 2 & \langle \alpha_2, \alpha_3 \rangle & \cdots & \langle \alpha_2, \alpha_\ell \rangle \\ \langle \alpha_3, \alpha_1 \rangle & \langle \alpha_3, \alpha_2 \rangle & 2 & \cdots & \langle \alpha_3, \alpha_\ell \rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle \alpha_\ell, \alpha_1 \rangle & \langle \alpha_\ell, \alpha_2 \rangle & \langle \alpha_\ell, \alpha_3 \rangle & \cdots & 2 \end{bmatrix}$$

(The diagonal is 2 since $\langle \alpha_i, \alpha_i \rangle = 2 \frac{(\alpha_i, \alpha_i)}{(\alpha_i, \alpha_i)} = 2$ for all i.)

Lemma 7.13. *For any $\alpha, \beta \in R$ and $g \in W$,*

$$\big\langle g(\alpha), g(\beta) \big\rangle = \langle \alpha, \beta \rangle$$

Proof. For any $\alpha, \beta, \gamma \in R$, since orthogonal transformations like reflections preserve the inner product,

$$\begin{aligned} \big\langle \sigma_\gamma(\alpha), \sigma_\gamma(\beta) \big\rangle &= 2 \frac{\big(\sigma_\gamma(\alpha), \sigma_\gamma(\beta)\big)}{\big(\sigma_\gamma(\beta), \sigma_\gamma(\beta)\big)} \\ &= 2 \frac{(\alpha, \beta)}{(\beta, \beta)} \\ &= \langle \alpha, \beta \rangle \end{aligned}$$

Since W is generated by the reflections σ_β for $\beta \in R$, any $g \in W$ is a composition of such reflections, so $\big\langle g(\alpha), g(\beta) \big\rangle = \langle \alpha, \beta \rangle$. ■

Lemma 7.14. *The Cartan matrix of a root system is unique up to reordering.*

Example. (*i*) For the root system A_2, we can take the base $\{\alpha,\beta\}$. Since the angle between them is $\frac{\pi}{2}$, $\langle\beta,\alpha\rangle = \langle\alpha,\beta\rangle = -1$. So, the Cartan matrix is

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

(*ii*) For the root system B_2

(*iii*)

(*iv*)

(*v*)

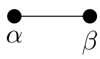
△

We can encode the information of the Cartan matrix in a graph as follows.

The *Dynkin diagram* of a root system R it the graph $\Delta = \Delta(R)$ with vertex set given by the base $\mathcal{B}$, and the number of edges between $\alpha,\beta \in \mathcal{B}$ given by $d_{\alpha,\beta} := \langle\alpha,\beta\rangle\langle\beta,\alpha\rangle$. If $d_{\alpha,\beta} > 1$, then the two roots α and β must have different lengths, and we notate the edges with an arrow pointing from the longer root to the shorter one.

Example.

(*i*) From the Cartan matrix of A_2, the Dynkin diagram is



(*ii*) From the Cartan

(*iii*)

(*iv*)

(*v*)

△

Note that the Dynkin diagram of a root system R can be constructed just from the Cartan matrix of R, and conversely, the Cartan matrix can be reconstructed from the Dynkin diagram, since $d_{\alpha,\beta}$ determines $\langle\alpha,\beta\rangle$ and $\langle\beta,\alpha\rangle$ for all $\alpha,\beta \in \mathcal{B}$.

Theorem 7.15. *Let R and R' be root systems of E and E' respectively. Then, $R \cong R'$ if and only if $\Delta(R) = \Delta(R')$.*

Lemma 7.16. *A root system R is irreducible if and only if $\Delta(R)$ is connected.*

Theorem 7.17. *Let R be an irreducible root system with Dynkin diagram $\Delta(R)$. Then, $\Delta(R)$ is one of the following types:*

(*i*) A_n, *$n \geq 1$;*

(*ii*) B_n, *$n \geq 2$;*

(*iii*) C_n, *$n \geq 3$;*

(*iv*) D_n, *$n \geq 4$;*

(*v*) G_2*;*

(*vi*) F_4*;*

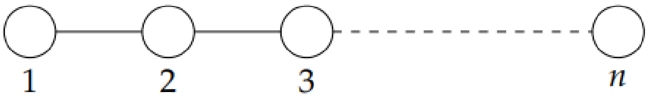
(*vii*) E_6*;*

(viii) E_7;

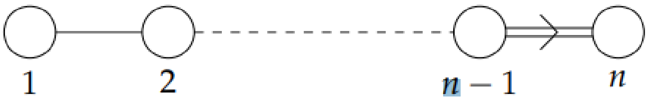
(ix) E_8.

Conversively, each such type occurs as the Dynkin diagram of a root system.

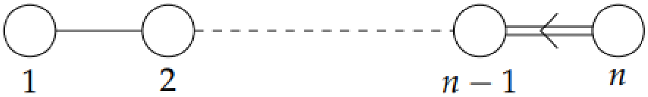
$A_n (n \geq 1)$:



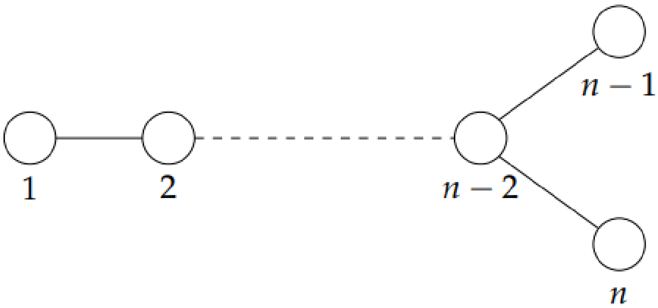
$B_n (n \geq 2)$:



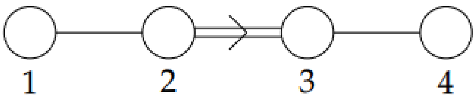
$C_n (n \geq 3)$:



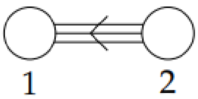
$D_n (n \geq 4)$:



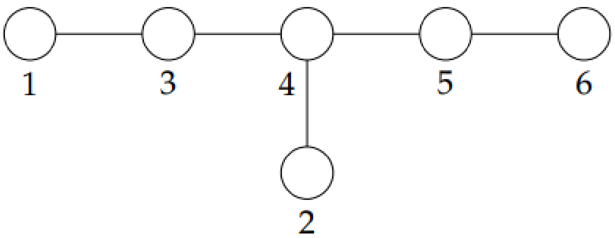
F_4 :



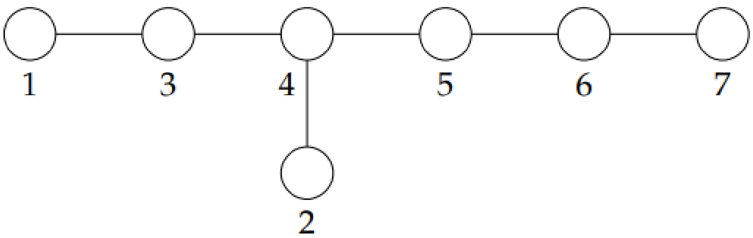
G_2 :



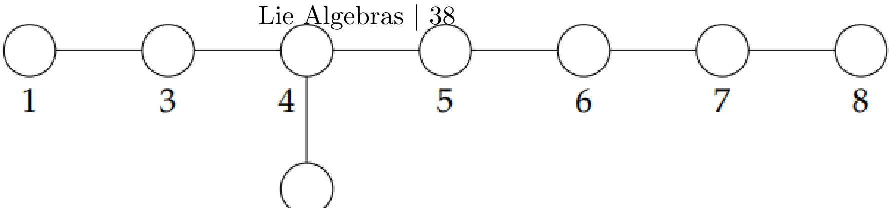
E_6 :



E_7 :



E_8 :



7.6 The Classification of Semisimple Complex Lie Algebras

Theorem 7.18. *Let L be a complex semisimple Lie algebra. If Φ_1 and Φ_2 are root system associated to two Cartan subalgebras of L, then $\Phi_1 \cong \Phi_2$.*

Corollary 7.18.1. *If L_1 and L_2 are complex semisimple Lie algebras with root systems Φ_1 and Φ_2 respectively, then $\Phi_1 \ncong \Phi_2$ implies $L_1 \ncong L_2$.*

Root System	Rank	Lie Algebra	Dimension	Number of Roots
A_n $(n \geq 1)$	n	$\mathfrak{sl}_{n+1}(\mathbb{C})$	$n^2 + 2n$	$n^2 + n$
B_n $(n \geq 2)$	n	$\mathfrak{so}_{2n+1}(\mathbb{C})$	$n(2n+1)$	$2n^2$
C_n $(n \geq 3)$	n	$\mathfrak{sp}_{2n}(\mathbb{C})$	$n(2n+1)$	$2n^2$
D_n $(n \geq 4)$	n	$\mathfrak{so}_{2n}(\mathbb{C})$	$n(2n-1)$	$n(2n-2)$
G_2	2	$\mathfrak{g}_2(\mathbb{C})$	14	12
F_4	4	$\mathfrak{f}_4(\mathbb{C})$	52	48
E_6	6	$\mathfrak{e}_6(\mathbb{C})$	78	72
E_7	7	$\mathfrak{e}_7(\mathbb{C})$	133	126
E_8	8	$\mathfrak{e}_8(\mathbb{C})$	248	240

Examples. We can use this information to determine the isomorphism classes of semisimple complex Lie algebras of each dimension.

1. There is no semisimple complex Lie algebra of dimension $1, 2, 4, 5$ or 7. Indeed, such a Lie algebra is the direct sum of simple Lie algebras and so its dimension is the sum of dimensions of simple Lie algebras. We have already seen that 3 is the smallest such dimension. The table shows that the next largest dimension is 8. It is now straightforward to check that $4, 5$ and 7 are impossible since they are not multiples of 3.

2. Since $6 = 3 + 3$, we see that 6 is a possible dimension. And there is a unique semisimple complex Lie algebra of dimension 6, namely $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ which has root system $A_1 \sqcup A_1$.

3. Similarly $9, 10$ and 11 are all possible. Indeed, we have $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{so}_5(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C})$.